Analysis of M $^{(x)}$ /G/1 with second optional service Interruption of three phase Vacations under Bernoulli Schedule

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Abstract : In this model, we study a batch arrival queueing system with second optional service interruption of three phase vacation based on Bernoulli schedule. Batches arrives in poission with mean arrival rate (λ >0), such that all customers demand the first essential service, where some of them demand the second optional service. The service times of the first essential service and the second optional service are assumed to follow general (arbitrary) distribution with distribution function B1(v)and B_2(v)respectively. After every service completion the server has the option to leave for phase one vacation of random length with probability p or to continue staying in the system with probability 1-p. As soon as the completion of phase one vacation, the server undergoes phase two and phase three vacations. On completion of three heterogeneous phase of vacation the server return back to the system. The vacation times are assumed to be general. The server is interrupted at random follows exponential distribution. Also we assume, the customer whose service is interrupted goes back to the head of the queue where the arrivals are Poisson. The time dependent probability generating function have been obtained interms of Laplace transform and corresponding steady state results have been derived explicity.Mean queue length and mean waiting time are also derived.

Keywords: Batch arrival, Probability Generating Function, Service Interruption,Three phases of vacation .

1. INTRODUCTION

Queuing systems with server vacations and/or random system breakdowns have been studied by numerous researchers including the survey of Doshi [5], Kulkarni(13) and Choi [13], Takagi, Takine and Sengupta [30], Wang et al, Madan etc.[9], Tian and Zhang, Maraghi et al and Thangaraj and Vanitha. However, in these models, the server stops the original work in the vacation period and can not come back to the regular busy period until the vacation period ends. In queueing theory periods of temporary service unavailability are referred to as server vacations, server interruptions or server breakdowns.

Queueing models with service interruptions have proved to be a useful abstraction in situations where a service facility is shared by multiple queues or where the facilities subject to failure. White and Christie have studied queues with service interruptions. They consider an M/M/1 queueing system with exponentially distributed interruptions. Generally distributed service times and interruptions are considered by Avi-Itzhak and Naor Vacation queues with c servers have been studied by Tian et al. . Borthakur and Choudhury [8] have studied vacation queues with batch arrivals. We assume that the customers arrive to the service station in batches of variable size, but are served one by one. We assume that the service times, vacation times, have a general distribution while the time to interruptions is exponentially distributed. Most of the recent studies have been devoted to batch arrival. In this paper, we consider a batch arrival queueing system $M^{[X]/}G/1$ with service interruption, in which we assume that after every service completion, the server has the option to leave for a vacation of random length with probability or to continue staying in the system with probability 1-p.The vacation period has three heterogeneous phases. On completion of three vacation phases the server return back to the system. Also we assume, the customer whose service is interrupted goes back to the head of the queue where the arrivals are Poisson. This paper is organized as follows. The mathematical description of our model is given in section 2. Definitions and Equations Governing the system is given in section 3. The time dependent solution has been obtained. Section 4 and corresponding steady state results have been derived.

2. MATHEMATICAL DESCRIPTION MODEL

We assume the following to describe the queueing model of our study.

(a).Customers arrive at the system in batches of variable size in a compound Poisson process and they are provided one by one service on a first come - first served basis. Let $\lambda c_i\, dt$ $(i = 1, 2, . ..)$ be the first order probability that a batch of ‘ i’ customers arrives at the system during a short interval of time (t; t + dt], where $0 \leq c_i \leq 1$ and $_{i=1}^{\infty} c_i = 1$ and $\lambda > 0$ is the arrival rate of batches.

b) A single server provides service to all arriving customer, with the service time having general distribution. Let B(v) and b(v)be the distribution and the density function of the service time respectively.

c) As soon as the first service of a customer is completed ,then he demand for the second service with probability r , or else he may decide to leave the system with probability (1-r) with out getting optional service in which case another customer at the head of the queue (if any) is taken up for his first essential service.

d) we assume interruptions arrive at random while serving the customers and assumed to occur according to a Poisson process with mean rate $\alpha > 0$. Let β be the server rate of attending interruption. Further we assume that once the interruption arrives the customer whose service is interrupted comes back to the head of the queue. Let μ(x)dx be the conditional probability of completion of the service during the interval (x; x + dx] given that the elapsed time is x, so that

$\mu(x) = \frac{b(x)}{1-B(x)}$ and therefore $b(s)=\mu(s)\, ce^{-{}_0^{\infty} \mu\ x\ d(x)}$,

d) As soon as the service is over, the server may take a vacation with probability p or may continue staying in the system with probability 1-p. After phase one vacation completion the server undergoes phase two and phase three vacations. On completion of three phase of vacation the server return back to the system.

e) The server's vacation time follows a general (arbitrary) distribution with distribution function $V_i(t)$ and density function $v_i(t)$. Let $\gamma_i(x)$dx be the conditional probability of a completion of a vacation during the interval (x; x + dx] given that the elapsed vacation time is x, so that

$$\gamma_i(x) = \frac{v_{i(x)}}{1-V_{i(x)}} \quad i = 1,2,3$$

and fore $v_i(t)=\gamma_i(t)\, e^{-{}_0^{s}\gamma_i\ x\ dIx)}$

f) On returning from vacation the server instantly starts serving the customer at the head of the queue if any.

3. DEFINITIONS AND EQUATIONS GOVERNING THE SYSTEM

(i) $P_n^i(x, t)$ = probability that at time 't' the server is active providing j^{th} service $j = 1,2$ and there are 'n' (n≥1) customers in the queue including the one being served and the elapsed service time for this customer is x. Consequently $P_n^i(t)$ denotes the probability that at time 't' there are 'n' customers in the queue excluding the one customer in i^{th} service irrespective of the value of x.

(ii) $V_n(x,t)$ = probability that at time 't', the server is under vacation with elapsed vacation time x, and there are 'n'(n≥1) customers waiting in the queue for service. Consequently $V_n^i(t)$

$= \int_0^\infty V_n^i \; x,t \; dt \,, i =$ 1,2,3 denotes the probability that at time 't' there are 'n' customers in the queue and the server is on vacation irrespective of the value of x.

(iii) $R_n(t)$=Probability that at time t, the server is inactive due to the arrival of interruption.

(iv) Q(t) = probability that at time 't' there are no customers in the system and the server is idle but available in the system .

The queueing model is then, governed by the following set of differential-difference equations:

$$\frac{\partial}{\partial x} P_n^{(1)}(x,t) + \frac{\partial}{\partial t} P_n^{(1)}(x,t) + (\lambda+\mu_1(x) + \alpha)\, P_n^{(1)}(x,t) = \lambda \sum_{k=1}^{n} c_k\, P_{n-k}^{(1)}(x,t) \;; n\geq 1 \tag{5}$$

$$\frac{\partial}{\partial x} P_0^{(1)}(x,t) + \frac{\partial}{\partial t} P_0^{(1)}(x,t) + (\lambda+\mu_1(x) + \alpha)\, P_0^{(1)}(x,t) = 0 \tag{6}$$

$$\frac{\partial}{\partial x} P_n^{(2)}(x,t) + \frac{\partial}{\partial t} P_n^{(2)}(x,t) + (\lambda+\mu_2(x) + \alpha)\, P_n^{(2)}(x,t) = \lambda \sum_{k=1}^{n} c_k\, P_{n-k}^{(2)}(x,t) \;; n\geq 1 \tag{7}$$

$$\frac{\partial}{\partial x} P_0^{(2)}(x,t) + \frac{\partial}{\partial t} P_0^{(2)}(x,t) + (\lambda+\mu_2(x) + \alpha)\, P_0^{(2)}(x,t) = 0 \tag{8}$$

$$\frac{\partial}{\partial x} V_n^{(i)}(x,t) + \frac{\partial}{\partial t} V_n^{(i)}(x,t) + (\lambda+\gamma_i(x))\, V_n^{(i)}(x,t) = \lambda \sum_{k=1}^{n} c_k\, V_{n-k}^{(i)}(x,t) \;; n\geq 1, \; i=1,2,3 \tag{9}$$

$$\frac{\partial}{\partial x} V_0^{(i)}(x,t) + \frac{\partial}{\partial t} V_0^{(i)}(x,t) + (\lambda+\gamma_i(x))\, V_0^{(i)}(x,t) = 0 \qquad i = 1,2,3 \tag{10}$$

$$\frac{d}{dt} R_0(t) = -(\lambda+\beta) R_0(t) \tag{11}$$

$$\frac{d}{dt} R_n(t) = -(\lambda+\beta) R_n(t) + \lambda \sum_{k=1}^{n} c_k\, R_{n-k}^{(i)}(t) + \alpha \int_0^\infty P_{n-1}^{(1)}(x,t)dx + \alpha \int_0^\infty P_{n-1}^{(2)}(x,t)\, dx \tag{12}$$

$$\frac{d}{dt} Q(t) = -\lambda Q(t) + \beta R_0(t) + \int_0^\infty \gamma_3(x) V_0^{(3)}(x,t)\, dx + (1-p)(1-r) \int_0^\infty P_0^{(1)}(x,t)\, \mu_1(x)dx + (1-p)(1-r) \int_0^\infty P_0^{(2)}(x,t)\, \mu_2(x)\, dx \tag{13}$$

Equations (5) to (13) are to be solved subject to the following boundary conditions.

$$P_n^{1}\;0,t \;=\; \lambda C_{n+1} Q(t) + \beta R_{n+1}(t) + (1-p)(1-r) \int_0^\infty P_{n+1}^{(1)}(x,t)\mu_1(x)dx + (1-p) \int_0^\infty P_{n+1}^{(1)}(x,t)\mu_2(x)dx + \int_0^\infty V_{n+1}^{(3)}(t)\, \gamma_3(x)dx, \quad n\geq 1. \tag{14}$$

$$P_n^{2}(0,t) = r \int_0^\infty P_n^{1}(x,t)\, \mu_1(x)\, dx, \quad n\geq 0 \tag{15}$$

$$V_n^{1}\;0,t \;=\; P(1-r) \int_0^\infty P_n^{(1)}(x,t)\mu_1(x)dx + P \int_0^\infty P_n^{(2)}(t)\, \mu_2(x)dx, \quad n\geq 0. \tag{16}$$

$$V_n^{2}\;0,t \;=\; \int_0^\infty V_n^{(1)}(x,t)\gamma_1(x)dx, \quad n\geq 0. \tag{17}$$

$$V_n^{3}\;0,t \;=\; \int_0^\infty V_n^{(2)}(x,t)\gamma_2(x)dx, \quad n\geq 0. \tag{18}$$

We assume that initially there are no customers in the system and the server is idle. so the initial Conditions are with

$$Q(0)=1,\; V_n^{(j)}(0)= V_0^{(j)}=0,\; R_n(0) = 0 \text{ and } P_n^{(i)}=0,\; n\geq 0. \quad i=1,2,\; j=1,2,3. \tag{19}$$

4. THE TIME DEPENDENT SOLUTION

Generating function of the queue length ; we define the Probability generating functions as follows

$P_q^{(i)}$(x,z,t) = $\sum_{n=0}^{\infty} P_n^{(i)}$(x,t) Z^n; i=1,2

$P_q^{(i)}$(z,t) = $\sum_{n=0}^{\infty} P_n^{(i)}$(t) Z^n; i=1,2

$P_q^{(i)}$(x,z,t) = $\sum_{n=0}^{\infty} V_n^{(i)}$(x,t) Z^n,

$P_q^{(i)}$(z,t) = $\sum_{n=0}^{\infty} V_n^{(j)}$(t) Z^n,

for j= 1,2,3

(20)

R_q(z,t) = $\sum_{n=0}^{\infty} R_n(t) Z^n$

C(z) = $\sum_{n=1}^{\infty} C_n Z^n$

Which are Convergent inside the circle given by $|Z| \leq 1$ and define the laplace transform of a function f(t) as f (s)= $\int_0^{\infty} e^{-st}$f(t) dt, R(S) > 0

(21)

Taking Laplace transform of equation (5) to (18) and usind using (19), we get

$\frac{\partial}{\partial x} P_n^{\,1}$ (x,s) + (S+λ+μ_1(x)+α) $P_n^{\,1}$ (x,s) =λ $\sum_{k=1}^{n-1} C_k$ $P_{n-k}^{\,1}$(x,s), n≥ 1 (22)

$\frac{\partial}{\partial x} P_0^{\,1}$ (x,s) + (S+λ+μ_1(x)+α) $P_0^{\,1}$ (x,s) = 0 (23)

$\frac{\partial}{\partial x} P_n^{\,2}$ (x,s) + (S+λ+μ_2(x)+α) $P_n^{\,2}$ (x,s) =λ $\sum_{k=1}^{n-1} C_k$ $P_{n-k}^{\,2}$(x,s), n≥ 1 (24)

$\frac{\partial}{\partial x} P_0^{\,2}$ (x,s) + (S+λ+μ_2(x)+α) $P_0^{\,2}$ (x,s) = 0 (25)

$\frac{\partial}{\partial x} V_n^{\,1}$ (x,s) + (S+λ+γ_1(x)) $V_n^{\,1}$ (x,s) =λ $\sum_{k=1}^{n} C_k$ $V_{n-k}^{\,1}$(x,s), n≥ 1 (26)

$\frac{\partial}{\partial x} V_0^{\,1}$ (x,s) + (S+λ+γ_1(x)) $V_0^{\,1}$ (x,s) =0 (27)

$\frac{\partial}{\partial x} V_n^{\,2}$ (x,s) + (S+λ+γ_2(x)) $V_n^{\,2}$ (x,s) =λ $\sum_{k=1}^{n} C_k$ $V_{n-k}^{\,2}$(x,s), n≥ 1 (28)

$\frac{\partial}{\partial x} V_0^{\,2}$ (x,s) + (S+λ+γ_2(x)) $V_0^{\,2}$ (x,s) =0 (29)

$\frac{\partial}{\partial x} V_n^{\,3}$ (x,s) + (S+λ+γ_3(x)) $V_n^{\,3}$ (x,s) =λ $\sum_{k=1}^{n} C_k$ $V_{n-k}^{\,3}$(x,s), n≥ 1 (30)

$\frac{\partial}{\partial x} V_0^{\,3}$ (x,s) + (S+λ+γ_3(x)) $V_0^{\,3}$ (x,s) =0 (31)

(S+λ+β)R_n (S) = λ $\sum_{k=1}^{n} C_k R_{n-k}(s)$ + α $\int_0^{\infty} P_{n-1}^{(1)}$ (x,s)dx + α $\int_0^{\infty} P_{n-1}^{(2)}$ (x,s)dx (32)

(S+λ+β)R_0(S) = 0 (32a)

(S+λ)Q(S) = 1+βR_0(S)+(1-p) (1-r) $\int_0^{\infty} P_0^{(1)}(x,s)\mu_1(x)dx$+ (1-P) $\int_0^{\infty} P_0^{(2)}(x,s)\mu_2(x)dx$ + $\int_0^{\infty} V_0^{(3)}(x,s)\gamma_3(x)dx$ (33)

$P_n^{(1)}$(0,S) = λ$C_{n+1}Q$(S) + βR_{n+1}(S) + (1-r)(1-p) $\int_0^{\infty} P_{n+1}^{(1)}(x,s)\mu_1(x)dx$ + (1-p) $\int_0^{\infty} P_{n+1}^{(2)}(x,s)\mu_2(x)dx$+ $\int_0^{\infty} V_{n+1}^{(3)}(x,s)\gamma_3(x)dx$

,n≥ 1, (34)

$P_n^{(2)}(0,S) = r \int_0^\infty P_n^{(1)}(x,s)\mu_1(x)dx$,n≥ 0 (35)

$V_n^{1}(0,s) = p(1-r)\int_0^\infty P_n^{(1)}(x,s)\mu_1(x)dx + p\int_0^\infty P_n^{(2)}(x,s)\mu_2(x)dx$ (36)

$V_n^2(0,s) = \int_0^\infty V_n^{(1)}(x,s)\gamma_1(x)dx$,n≥ 0 (37)

$V_n^{3}(x,s) = \int_0^\infty V_n^{(2)}(x,s)\gamma_2(x)dx$,n≥ 0 (38)

Now multiplying (22) by z^n and summing over n from 1 to ∞ , adding to equation (23) and using the definition of probability generating function , we obtain

$\frac{\partial}{\partial x} P_q^{1}(x,z,s) + (S+\lambda-\lambda c(z)+\mu_1(x)+\alpha) P_q^{1}(x,z,s) = 0$ (39)

Performing similar operations on equation (24) to (33)

$\frac{\partial}{\partial x} P_q^{2}(x,z,s) + (S+\lambda-\lambda c(z)+\mu_2(x)+\alpha) P_q^{2}(x,z,s) = 0$ (40)

$\frac{\partial}{\partial x} V_q^{1}(x,z,s) + (S+\lambda-\lambda c(z)+\gamma_1(x)) V_q^{1}(x,z,s) = 0$ (41)

$\frac{\partial}{\partial x} V_q^{2}(x,z,s) + (S+\lambda-\lambda c(z)+\gamma_2(x)) V_q^{2}(x,z,s) = 0$ (42)

$\frac{\partial}{\partial x} V_q^{3}(x,z,s) + (S+\lambda-\lambda c(z)+\gamma_3(x)) V_q^{3}(x,z,s) = 0$ (43)

$(S+\lambda-\lambda c(z)+\beta)\, R(z,s) = \alpha z\int_0^\infty P_q^{(1)}(x,z,s)\,dx + \alpha z\int_0^\infty P_q^{(2)}(x,z,s)\,dx$ (44)

for the boundary condition multiply both sides of equation (34) by z^n , summing over 1 to ∞ by using the definition of probability generating function equation ,we get

$ZP_q^{(1)}(0,z,s) = (1-r)(1-p)\int_0^\infty P_q^{(1)}(x,z,s)\mu_1(x)dx + (1-p)\int_0^\infty P_q^{(2)}(x,z,s)\mu_2(x)dx + \int_0^\infty V_q^{(3)}(x,z,s)\gamma_3(x)dx + \lambda[C(z)-1]Q(S) + [1-SQ(S)] + \beta R(z,s)$

(45) Performing similar operation on equation (35) to (38) we obtain

$P_q^{(2)}(0,Z,S) = r\int_0^\infty P_q^{(1)}(x,z,s)\mu_1(x)dx$ (46)

$V_q^{(1)}(0,Z,S) = P(1-r)\int_0^\infty P_q^{(1)}(x,z,t)\mu_1(x)dx + P\int_0^\infty P_q^{(2)}(x,z,t)\mu_2(x)dx$, n≥ 0 (47)

$V_q^{(2)}(0,Z,S) = \int_0^\infty V_q^{(1)}(x,z,s)\gamma_1(x)dx$, (48)

$V_q^{(3)}(0,Z,S) = \int_0^\infty V_q^{(2)}(x,z,s)\gamma_2(x)dx$ (49)

Intergrating the equation (39) from 0 to x yields

$P_q^{(1)}(x,Z,S) = P_q^{(1)}(0,Z,S)\, e^{-(s+\lambda-\lambda C(z)+\alpha)x-\int_0^x \mu_1(t)dt}$ (50)

where $P_q^{(1)}(0,Z,S)$ is given by (45) , Again integerating equation (50) by parts w. r. to x yields

$P_q^{(1)}(x,Z,S) = P_q^{(1)}(0,Z,S)\,\frac{1-B1(s+\lambda-\lambda C(z)+\alpha)}{s+\lambda-\lambda C(z)+\alpha}$ (51)

where $B1(s+\lambda-\lambda C(z)+\alpha)=$

$e\int_0^{\infty} - s+\lambda-\lambda C\ z\ +\alpha\ x$ $dB_1(x)$ is the laplace transform of the first essential service B_1 (x) . Now multiplying both sides of equation (50) by $\mu_1(x)$ and integrating over x, we get

$$\int_0^{\infty} P_q^{1}\ x,z,s\ \mu_1\ x\ dx = P_q^{1}\ 0,z,s$$

$B_1(s+\lambda-\lambda C\ z\ +\ \alpha)$ (52)

Similarly as Integrating equation (40) to (43) from 0 to λ, we get

P_q^{2} (x,z,s) =

P_q^{2} (0,z,s)$e^{- s+\lambda-\lambda c\ z\ \ x-\int_0^x \mu_2\ t\ dt}$

(53) V_q^{1} (x,z,s) =

V_q^{1} (0,z,s)$e^{- s+\lambda-\lambda c\ z\ \ x-\int_0^x \gamma_1\ t\ dt}$

(54)

V_q^{2} (x,z,s)

=V_q^{2} (0,z,s) $e^{- s+\lambda-\lambda c\ z\ \ x-\int_0^x \gamma_2\ t\ dt}$

(55) V_q^{3} (x,z,s) =

V_q^{3} (0,z,s) $e^{- s+\lambda-\lambda c\ z\ \ x-\int_0^x \gamma_3\ t\ dt}$

(56) Again Integrating from (53) to (56)

$P_q^{(2)}$(Z,S)

=$P_q^{(2)}$(0,Z,S) $\frac{1-B_2(s+\lambda-\lambda C\ z\ +\alpha)}{s+\lambda-\lambda C\ z\ +\alpha}$

(57) $V_q^{(1)}$(Z,S)

=$V_q^{(1)}$(0,Z,S) $\frac{1-V_1(s+\lambda-\lambda C\ z\ )}{s+\lambda-\lambda C\ z}$

(58) $V_q^{(2)}$(Z,S)

=$V_q^{(2)}$(0,Z,S) $\frac{1-V_2(s+\lambda-\lambda C\ z\ )}{s+\lambda-\lambda C\ z}$

(59) $V_q^{(3)}$(Z,S)

=$V_q^{(3)}$(0,Z,S) $\frac{1-V_3(s+\lambda-\lambda C\ z\ )}{s+\lambda-\lambda C\ z}$

(60) Where V_1(s+λ-λC(z))= $e\int_0^{\infty} - s+\lambda-\lambda C\ z\ \ x$ $dV_1(x)$

(61)

V_2(s+λ-λC(z))= $e\int_0^{\infty} - s+\lambda-\lambda C\ z\ \ x$ dV_2 (x) (62) V_3(s+λ-λC(z))= $e\int_0^{\infty} - s+\lambda-\lambda C\ z\ \ x$ dV_3 (x)

(63)

is the laplace – stielfjes transform of the first phase, second phase and third phase of varation time V_1 (x), V_2 (x),and V_3 (x) respectively Now multiplying both sides of equations (53) by $\mu_2(x)$ and integrating over x, we get

$\int_0^{\infty} P_q^{(1)}(x,z,s)\mu_2(x)dx = P_q^{(1)}(0,z,s)$

B_1(s+λ-λc(z)+α) (64)

Now using equation (52), equation (46) reduces to

$P_q^{(2)}$(0,z,s) = r$P_q^{(2)}$(0,z,s) B_2(s+λ-λc(z)+α) (65)

Now multiplying both sides of equations (54), (55), (56) by γ1(x), γ2(x) and γ3(x), integrating over x we obtain

$\int_0^{\infty} V_q^{(1)}(x,z,s)\ \gamma_1(x)dx = V_q^{(1)}(0,z,s)$

V_1(s+λ-λc(z)) (66)

$\int_0^{\infty} V_q^{(2)}(x,z,s)\ \gamma_2(x)dx = V_q^{(2)}(0,z,s)$

V_2(s+λ-λc(z)) (67)

$\int_0^{\infty} V_q^{(3)}(x,z,s)\ \gamma_3(x)dx = V_q^{(3)}(0,z,s)$

V_3(s+λ-λc(z)) (68)

the equation (57) becomes

$P_q^{(2)}$(z,s) = r $P_q^{(1)}$(0,z,s) $\frac{B_1\ s+\lambda-\lambda c\ z\ +\alpha\ (1-B_2\ (s+\lambda-\lambda c\ z\ +\alpha)}{s+\lambda-\lambda c\ z\ +\alpha}$

(69)

Now using equation (52), (64), (65) equation (47)

$V_q^{(1)}$(0,z,s) = p(1-r) $P_q^{(1)}$(0,z,s) B_1(s+λ-λc(z)+α) + p r $P_q^{(1)}$(0,z,s) B_1(s+λ-λc(z)+α) (70)

Using above equation (70), equation (58),(59),and (60) becomes

$$V_q^{(1)}(z,s) = P_q^{(1)}(0,z,s)p \quad 1 - r \quad B_1 \ f1(z) \quad + rB_1 \ f1(z) \quad B_2 \ f1(z) \quad \frac{1 - V_1(s+\lambda-\lambda c(z))}{(s+\lambda-\lambda c(z))} \qquad (71)$$

$$V_q^{\ 2}(z,s) = P_q^{\ 1}(0,z,s)p \quad 1 - r \ B_1 \ f1 \ z \quad + rB_1 \ f1 \ z \quad B_2 \ f1 \ z \quad V_1(s+\lambda-\lambda c(z)) \ \frac{1 - V_2(s+\lambda-\lambda c(z))}{(s+\lambda-\lambda c(z)} \qquad (72)$$

$$V_q^{\ 3}(z,s) = P_q^{\ 1}(0,z,s)p \quad 1 - r \ B_1 \ f1 \ z \quad + rB_1 \ f1 \ z \quad B_2 \ f1 \ z \quad V_1(s+\lambda-\lambda c(z))V_2(s+\lambda-\lambda c(z)) \ \frac{1 - V_3(s+\lambda-\lambda c(z))}{(s+\lambda-\lambda c(z)} \qquad (73)$$

where $f_1(z) = s+\lambda-\lambda c(z)+\alpha$. using equation (51), (64) and (65), equation (44) becomes

$$R_q(z,s) = \frac{\alpha z P_q^{(1)}(0,z,s) \ 1- \ 1-r \ B_1 \ f1 \ z \ +rB_1 \ f1 \ z \quad B_2 \ f1 \ z}{f1(z) \quad f2(z)} \qquad (74)$$

where $f_2(z) = s+\lambda-\lambda c(z)+\beta$. Now using equation (52),(64),(66) and (740 in equation (45) and solving for $P_q^{(1)}(0,z,s)$,we get

$$P_q^{(1)}(0,z,s) = \frac{f1(Z) \ f2(Z) \quad \lambda \ C \ z \ -1 \ Q \quad s \quad +(1-sQ\ (s)}{Dr}$$

where

$$Dr = f_1(z) \ f_2(z) \ \{z - \quad 1-p \ - pV1(s+\lambda-\lambda c \ z \ V2(s+\lambda-\lambda c \ z \ V3(s+\lambda-\lambda c \ z \quad rB_1 \ f1(z) \quad B_2 \ f1(z) \ \}$$
$$- \alpha\beta z \ 1 - \ 1 - r \ B_1 \ f1(z) \ - rB_1 \ f1(z) \quad B_2 \ f1(z)$$

(76)

Substituting the value of $P_q^{(1)}(0,z,s)$ from equation (74) in to (51), (57), (58), (59) & 60

We get

$$P_q^1(z,s) = \frac{f_2(z) \ 1-B_1(f_1(z) \quad \lambda \ c(z-1)Q \ s \ +(1-sQ(s)}{Dr} \qquad (77)$$

$$P_q^2(z,s) = \frac{rf_2(z)B_1[f_1(z)] \ 1-B_2(f_1(z) \ \lambda(c \ z \ -1Q \ s \ +(1-sQ(s)]}{Dr} \qquad (78)$$

$$\boldsymbol{V_q^1}(z,s) = \frac{Pf_1f_2(z) \quad 1-r \ B_1 \ f_1 \ z \ +rB_1 \ f_1 \ z \quad B_2 \ f_1 \ z \quad \lambda(c \ z \ -1Q \ s \ +(1-sQ(s)}{Dr} \ \frac{1-V_1(s+\lambda-\lambda c(z)}{s+\lambda-\lambda c(z)} \qquad (79)$$

$$\boldsymbol{V_q^2}(z,s) = \frac{Pf_1f_2(z) \quad 1-r \ B_1 \ f_1 \ z \ +rB_1 \ f_1 \ z \quad B_2 \ f_1 \ z \quad \lambda(c \ z \ -1Q \ s \ +(1-sQ(s) \ V_1(s+\lambda-\lambda c(z))}{Dr} \ \frac{1-V_2(s+\lambda-\lambda c(z)}{s+\lambda-\lambda c(z)} \qquad (80)$$

$$\boldsymbol{V_q^3}(z,s) = \frac{Pf_1f_2(z) \quad 1-r \ B_1 \ f_1 \ z \ +rB_1 \ f_1 \ z \quad B_2 \ f_1 \ z \quad \lambda(c \ z \ -1Q \ s \ +(1-sQ(s) \ V_1(s+\lambda-\lambda c(z)V_2(s+\lambda-\lambda c(z))}{Dr} \ \frac{1-V_2(s+\lambda-\lambda c(z)}{s+\lambda-\lambda c \ z} \qquad (81)$$

5. THE STEADY STATE ANALYSIS

In this section, derive the steady state probability distribution for our queueing model. To define the steady state probabilities, suppress. The argument ‘t’ wherever it appears in the time dependent analysis

$$\lim_{s\to 0} sf \ s = \lim_{t\to\infty} f(t) \qquad (82)$$

Multiplying both sides of equation (77), (78), (79), (80), (81) by and applying Property (82) and simplifying, we get

$\boldsymbol{P_q^{(1)}}(z) = \frac{f_2(z)\ 1-B_1(f_1(z)\ \ \lambda\ c_z\ -1)Q}{Dr}$

(83) $\boldsymbol{P_q^{(2)}}(z)$

$== \frac{rf_2(z)B_1\ f_1(z)\ \ 1-B_1(f_1(z)\ \ \lambda\ c_z\ -1)Q}{Dr}$

(84)

$\boldsymbol{V_q^1}(z) =$

$\frac{Pf_1f_2\ z\ \ \ 1-r\ B_1\ f_1\ z\ \ +rB_1\ f_1\ z\ \ B_2\ f_1\ z\ \ \ [V1\ \lambda-\lambda c\ z\ )-1\ Q]}{Dr}$

(85) $\boldsymbol{V_q^2}(z) =$

$\frac{Pf_1f_2\ z\ \ \ 1-r\ B_1\ f_1\ z\ \ +rB_1\ f_1\ z\ \ B_2\ f_1\ z\ \ \ [V1\ \lambda-\lambda c\ z\ \ [V2\ \lambda-\lambda c\ z\ )-1\ Q]}{Dr}$

(86)

$V_q^3(z) =$

$\frac{Pf_1\ \ z\ f_2\ z\ \ \ 1-r\ B_1\ f_1\ z\ \ +rB_1\ f_1\ z\ \ B_2\ f_1\ z\ \ \ [V1\ \lambda-\lambda c\ z\ \ [V2\ \lambda-\lambda c\ z\ \ [V3\ \lambda-\lambda c\ z\ )-1\ Q]}{Dr}$

$R_q^-\ (z) =$

$\frac{\alpha z\ 1-\ 1-r\ B_1\ f_1\ z\ \ -rB_1\ f_1\ z\ \ B_1\ f_1\ z\ \ \ [\ \lambda c\ z\ )-1\ Q]}{Dr}$

(87)

Let $W_q(Z)$ denotes the PGF of queue size irrespective of the state of the system. Then adding (85), (86), (87), & (88)

$\mathbf{W_q(Z)} = P_q^1\ (z) + P_q^2\ (z) + V_q^1\ (z) + V_q^2\ (z) + V_q^3\ (z) + R_q^-\ (z)$ (88)

In order to obtain Q, using the normalization condition

$W_q(1) + Q = 1$

We see that for Z=1, $W_q(Z)$ is indeterminate of the frrm %. We apply L Hospital's rule in equ(88),. where Bi(0)=i=1,2,

(1)=a, $C_1^1(1)$=E(I) is mean batch size if the arriving customers

V(0)=1,$V_0^i = E$(v), E(vi)=i=1,2,3.. the mean vacation time

$P_q^{(1)}\ (1) = \frac{\lambda BQE(I)\ 1-B_1(\alpha)}{dr}$

(89)

$P_q^{(2)}\ (1) = \frac{r\lambda BQE\ I\ \ \ 1-B_2\ \alpha\ \ B_1\ \alpha}{dr}$

(90) $V_q^{(1)}\ (1) =$

$\frac{P\alpha E\ I\ \lambda BQ(I)\ \ 1-r\ B_1\ \alpha\ +rB_1\ \alpha\ +B_2\ \alpha\ \ \ E(V_1)}{dr}$

(91) $V_q^{(2)}\ (1) =$

$\frac{P\alpha E\ I\ \lambda BQ(I)\ \ 1-r\ B_1\ \alpha\ +rB_1\ \alpha\ +B_2\ \alpha\ \ \ E(V_2)}{dr}$

(92)

$V_q^{(3)}\ (1) =$

$\frac{P\alpha E\ I\ \lambda BQ(I)\ \ 1-r\ B_1\ \alpha\ +rB_1\ \alpha\ +B_2\ \alpha\ \ \ E(V_3)}{dr}$

(93)

$R_q\ (1) =$

$\frac{\alpha\lambda QE\ I\ \ \ 1-r\ B_1\ \alpha\ -rB_1\ \alpha\ B_2\ \alpha}{dr}$

(94)

Where dr = $\alpha\beta\ \ \ 1-r\ \ B_1\ \ \alpha\ \ + rB_1\ \ \alpha\ \ B_2\ \ \alpha\ \ \ -\lambda E\ \ I\ \ \ \alpha+\beta\ \ \ 1- \ \ 1-r\ \ B_1\ \ \alpha\ \ -rB_1\ \ \alpha\ \ B_2\ \ \alpha$ - λα β P E(I)E(V)[(1-r) $B_1\ \ \alpha\ \ + rB_1\ \ \alpha\ \ B_2\ \ \alpha$]

$W_q(1) = P_q^1\ (1) + P_q^2\ (1) + V_q^1\ (1) + V_q^2\ (1) + V_q^3\ (1) + R_q\ (1)$

$W_q^{(1)} =$

$\frac{\lambda E\ I\ \ \ \alpha+\beta\ \ \ 1-\ \ 1-r\ B_1\ \alpha\ +rB_1\ \alpha\ +B_2\ \alpha}{Dr} +$

$\frac{\lambda E\ I\ (\alpha+\beta)\ 1-\ \ 1-r\ B_1\ \alpha\ -rB_1\ \alpha\ +B_2\ \alpha\ \ \ E(V)}{Dr}$

(95)

Q = 1-λE(I)

$[\frac{1}{\beta(1-r)B_1\ \alpha\ +rB_1\ \alpha\ +B_2\ \alpha}]$

$+\frac{1}{\alpha(1-r)B_1\ \alpha\ +rB_1\ \alpha\ +B_2\ \alpha} - \frac{1}{\alpha} - \frac{1}{\beta} +$

PE(V)]. (96)

Q=1-ρ and the utilization factor ρ of the system is given by

ρ= λE(I) $[\frac{1}{\beta(1-r)B_1\ \alpha\ +rB_1\ \alpha\ B_2\ \alpha}] +$

$[\frac{1}{\alpha(1-r)B_1\ \alpha\ +rB_1\ \alpha\ +B_2\ \alpha} - \frac{1}{\alpha} - \frac{1}{\beta} +$

PE(V)]

Where ρ<1 is the stability condition under which the steady state exists, equation (96) gives the Probability that the server is idle.

6.THE AVERAGE QUEUE SIZE

Let Lq denote the mean number of customers in the queue under the steady state then $\frac{d}{dz}\ W_q^-\ z\ _{z=1}$ snice this gives % form we write $W_q^-\ z\ = \frac{N(z)}{D(z)}$ where N(Z) & D(Z) are the number and denominator of the right hand side of equation (88) respectively then we use $L_q = \lim_{z\to 1}[\frac{D^{'}N^{''}-D^{''}N^{'}}{2(D^{'})^2}]$ (97)

where primes and double primes in equation (97) denote first and second derivation at z=1 respectively

$N(z) = \lambda E\ I\ \ \alpha+\beta\ \ 1- \ 1-r\ B_1\ \alpha\ + rB_1\ \alpha\ B_2\ \alpha\ + \alpha\beta\lambda E\ V\ E\ I\ \ 1-r\ B_1\ \alpha\ - rB_1\ \alpha\ B_2\ \alpha$

$D(z) = \alpha\beta\ \ 1-r\ B_1\ \alpha\ + rB_1\ \alpha\ B_2\ \alpha\ + \ 1-\lambda E\ I\ PE\ V\ - \lambda E\ I\ (\alpha+\beta)\ 1-\ \ 1-r\ B_1\ \alpha\ - rB_1\ \alpha\ B_2\ \alpha$

we have $N'\ 1\ = Q\lambda E(I)\{(\alpha+\beta) + ((1-r)B_1(\alpha) + rB_1(\alpha)B_2(\alpha))[p\,\alpha\,\beta\,E(V) - \alpha - \beta]$ (98)

$$N"\ 1\ = 2Q[\lambda E(I)]\,\{(\frac{\alpha}{\lambda E(I)} - 1) + (1-r)B_1(\alpha) + r\,B_1(\alpha)B_2(\alpha))[1 - \frac{\alpha}{\lambda E(I)} - p(\alpha+\beta)E(V) + \frac{1}{2}p\alpha\beta E(V^2)] + (1-r)B_1(\alpha) + rB_1(\alpha) + B_2(\alpha))[\alpha+\beta - p\alpha\beta(V)] + \lambda E(I-1))Q\ \ \alpha+\beta\ + \ 1-r\ B_1\ \alpha\ + rB_1\ \alpha\ B_2\ \alpha\ \ p\alpha\beta E\ V\ - \alpha-\beta$$ (99)

$D^{'}\ 1\ = -\lambda E\ I\ \ \alpha+\beta\ \ 1-r\ B_1\ \alpha\ + rB_1\ \alpha\ B_2\ \alpha\ \ \alpha\beta + \ \lambda E\ I\ \ \alpha+\beta\ - p\alpha\beta E\ V$

(100)

$D"\ (1) = 1 - \frac{\alpha+\beta}{\lambda E\ I} + (\ 1-r\ B_1\ \alpha\ + r\,B_1\ \alpha\ B_2\ \alpha\ )\ -1 + p\ \alpha+\beta\ E\ V\ - \frac{1}{2}p\alpha\beta E\ V2\ + ((1-r))B_1(\alpha) + rB_1(\alpha) + B_2(\alpha))[-(\alpha+\beta)\ \frac{\alpha\beta}{\lambda E\ I}p\alpha\beta E(V)] + \lambda E(I(I-1)) - [-(\alpha+\beta) + (\ 1-r\ B_1\ \alpha\ + rB_1\ \alpha\ + B_2\ \alpha\ )[\alpha+\beta - p\alpha\beta E(V)]$

(101)

where $E(V_2)$ is the second moment of the vacation time and Q has been found in equation (96). Then if we substitute the values of N ′ (1), N" (1), D′ (1) and D" (1) from equations (98),(99),(100),(101) in to equation(97). we obtain L_q in a closed form. Mean waiting time of a customer could be found as

$w_q = \frac{Lq}{\lambda}$ by using Little's formula.

7. CONCLUSION

In this paper we have studied a batch arrival, essential service with interruption and three phases of vacation. This paper clearly analyzes the transient solution, steady state results of our queueing system.As a future work busy period analysis and reliability analysis will be determined.

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