$M^{[X]}$/G/1 with Second Optional Service Multiple Vacation, Breakdown and Repair

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Abstract : In this paper, we present a batch arrival non- Markovian queuing model with second optional service, subject to random break downs and multiple vacations. Batches arrive in Poisson stream with mean arrival rate, such that all customers demand the first `essential' service, whereas only some of them demand the second `optional' service. The service times of the both first essential service and the second optional service are assumed to follow general (arbitrary) distribution with distribution function $B_1(v)$ and $B_2(v)$ respectively. The server may undergo breakdowns which occur according to Poisson process with breakdown rate. Once the system encounter break downs it enters the repair process and the repair time is followed by exponential distribution with repair rate α. The server takes vacation each time the system becomes empty and the vacation period is assumed to be exponential distribution. On returning from vacation if the server finds no customer waiting in the system, then the server again goes for vacation until he finds at least one unit in the system. The time-dependent probability generating functions have been obtained in terms of their Laplace transforms and the corresponding steady state results have been derived explicitly. Also the mean queue length and the mean waiting time have been found explicitly.

Keywords: $M^{[X]}/G/1$ Queue, First essential service, Second optional service, Multiple vacations, Probability generating function, Transient state, Steady state.

SUBJECT CLASSIFICATION :AMS 60K25, 60K30

1. INTRODUCTION

The research study on queuing systems with server vacation has become an extensive and interesting area in queuing theory literature. Server vacations are used for utilization of idle time for other purposes. Vacation queuing models has been modeled effectively in various situations such as production, banking service, communication systems, and computer networks etc. Numerous authors are interested in studying queuing models with various vacation policies including single and multiple vacation policies. Batch arrival queue with server vacations was investigated by Yechiali (1975). An excellent comprehensive studies on vacation models can be found in Takagi (1991) and Doshi (1986) research papers. One of the classical vacation model in queuing literature is Bernoulli scheduled server vacation. Keilson and Servi(1987) introduced and studied vacation scheme with Bernoulli schedule discipline. Madan(2001) studied queue system with compulsory vacation in which the server should go for vacation with probability 1 whenever the system becomes empty. Later on, the same author discussed many queuing models with Bernoulli scheduled server vacation Baba(1986) employed the supplementary variable technique for deriving the transform solutions of waiting time for batch arrival with vacations. Relating with the server of a queuing system, the server may be assumed as a reliable one, but this is not the case in most of real scenarios that the server will not be last forever in order to provide service. So in this context, numerous papers of the server may be assumed unreliable, which can encounter breakdown. Thus queuing model with server break down is a remarkable and unavoidable phenomenon and the study of queues with server breakdowns and repairs has importance not only in the point of theoretic view but also in the engineering applications. Avi-Itzhak(1963) considered some queuing problems with the servers subject to breakdown. Kulkarni et al.(1990) studied retrial queues with server subject to breakdowns and repairs. Tang(1997) studied M/G/1 model with server break down and discussed reliability of the system. Madan et al.(2003) obtained the steady state results of single server Markovian model with batch service subject to queue models with random breakdowns. Queuing systems with random break downs and vacation have also been keenly analyzed by many authors including Grey (2000) studied vacation queuing model with service breakdowns. Madan and Maraghi (2009) have obtained steady state solution of batch arrival queuing system with random breakdowns and Bernoulli schedule server vacations having general vacation time. Thangaraj(2010) studied the transient behaviour of single server with compulsory vacation and random break downs.

Queuing models with Second optional service plays a prominent role in the research study of queuing theory. In this type of queuing model, the server performs first essential service to all arriving customers and after completing the first essential service, second optional service will be provided to some customers those who demand a second optional service. Madan(2000) has first introduced the concept of second optional service of an M/G/1 queuing system in which he has analyzed the time-dependent as well as the steady state behaviour of the model by using supplementary variable technique. Medhi(2001)proposed an M/G/1 queuing model with second optional channel who developed the explicit expressions for the mean queue length and mean waiting time. Later Madhan(2002) studied second optional service byincorporating Bernoulli schedule server vacations. Gaudham. Choudhury(2003) analyzed some aspects of M/G/1 queuing system with second optional service and obtained the steady state queue size distribution at the stationary point of time for general second optional service. A batch arrival with two phase service model with re-service for each phase of the service has been analyzed by Madan et al.(2004). Wang (2004) studied an M/G/1 queuing system with second optional service and server breakdowns based on supplementary variable technique. Kalyanaraman et al.(2008) studied additional optional batch service with vacation for single server queue.

In this paper we consider queuing system such that the customers are arriving in batches according to Poisson stream. The server provides a first essential service to all incoming customers and a second optional service will be provided to only some of them those who demand it. Both the essential and optional service times are assumed to follow general distribution. The vacation times and the repair time are exponentially distributed. Whenever the system meets a break down, it enters in to a repair process and the customer whose service is interrupted goes back to the head of the queue. Customers arrive in batches to the system and are served on a first come-first served basis.

The rest of the paper is organized as follows. The mathematical description of our model is in Section 2 and equations governing the model are given in Section 3. The time dependent solutions have been obtained in Section 4 and the corresponding steady state results have been derived explicitly in Section 5.

2. MATHEMATICAL DESCRIPTION OF THE MODEL

The following assumptions are to be used describe the mathematical model of our study:

- Customers arrive at the system in batches of variable size in a compound Poisson process and they are provided service one by one on a 'first come first served' basis. Let $\Sigma C_k \, dt \; (k = 1, 2, 3, \ldots)$ be the first order probability that a batch of k customers arrives at the system during a short interval of time (t, t + dt], $where \; 0 \leq C_k \leq 1 \; and \; \Sigma \, C_k = 1 \; and \; \lambda > 0$ is the mean arrival rate of batches. Let ; k = 1, 2, 3, ... be the first order probability of arrival of 'k' customers in batches in the system during a short period of time (t, t+dt) $where \; 0 \leq C_k \leq 1; \quad C_k = 1, \lambda > 0$ is the mean arrival rate of batches.

- There is a single server which provides the first essential service to all arriving customers. Let $B_1(v) \; and \; b_1(v)$ respectively be the distribution function and the density function of the first service times respectively.

- As soon as the first service of a customer is completed, then he may demand for the second service with probability r, or else he may decide to leave the system with probability 1- r in which case another customer at the head of the queue (if any) is taken up for his first essential service.

- The second service times as assumed to be general with the distribution function $B_2(v)$ and the density function $b_2(v)$. Further, Let $\mu_i(x)dx$ be the conditional probability density function of ith service completion during the interval (x, x+dx] given that the elapsed service time is x, so that

- If there is no customer waiting in the queue, then the server goes for a vacation. The vacation periods are exponentially distributed with mean vacation time $\frac{1}{\gamma}$. On returning from vacation if the server again founds no customer in the queue, then it goes for another vacation. So the server takes multiple vacations

- The system may break down at random and breakdowns are assumed to occur according to a Poisson stream with mean breakdown rate α > 0:

- Once the system breaks down, it enters a repair process immediately. The repair times are exponentially distributed with mean repair rate β > 0.

- Various stochastic processes involved in the system are independent of each others.

3. DEFINITIONS AND EQUATIONS GOVERNING THE SYSTEM

- $P_n^i(x,t)$ = probability that at time 't' the server is active providing ith service and there are 'n' customers in the queue including the one being served and the elapsed service time for this customer is x. Consequently $P_n^i(t)$ = denotes the probability that at time 't' there are 'n' customers in the queue excluding the one customer in ith service irrespective of the value of x.

- $V_n(t)$ = the probability that at time 't' there are 'n' customers in the queue and the server is on vacation irrespective of the value of x.

- $R_n(t)$ = Probability that at time t, the server is inactive due to break down and the system is under repair while there are 'n' () customers in the queue.

The queuing model is then, governed by the following set of differential-difference equations:
Let $NQ(t)$ denote the queue size(excluding one in service) at time t. We introduce the random variable Y(t) as follows

$$Y(t) = \begin{cases} 1, \textit{if the system is busy with } \text{first essential service } \textit{at time } t \\ 2, \textit{if the system is busy with second } \text{ service } \textit{at time } t \\ 3, \textit{if the system is on vacation at time } t \end{cases}$$

We introduce the supplementary variable as,

$$L(t) = \begin{cases} B_1^0(t) & if\ Y(t) = 1 \\ B_2^0(t) & if\ Y(t) = 2 \\ V^0(t) & if\ Y(t) = 3 \end{cases}$$

where

$B_1^0(t)$ = elapsed service time for the first essential service at time t,
$B_2^0(t)$ = elapsed service time for the second service at time t,
$V^0(t)$ = elapsed vacation time of the server at time t.

The process $\{NQ(t), L(t)\}$ is a continuous time Markov process. we define the probabilities for i = 1,2.

$$P_n^i(x,t) = Prob\{NQ(t) = n, L(t) = B_i^0; x < B_i^0 \le x + dx;\ x > 0, n > 0\}$$

In steady state condition, we have

$P_n^i(x)\,dx = \lim_{t\to\infty} P_n^i(x,t)$, i=1, 2 $x > 0$; $n \geq 0$

$V_n = \lim\limits_{t\to\infty} V_n(t)$; $n \geq 0$

$R_n = \lim\limits_{t\to\infty} R_n(t)$; $n \geq 0$

Assume that

$V_0(0) = 1, V_n(0) = 0$

And for i=1,2

$B_i(0) , B_i(\infty) = 1$
Also V(x) and $B_i(x)$ are continuous at $x = 0$.

The model is then, governed by the following set of differential-difference equations:

$$\frac{\partial}{\partial x}P_n^{1}(x,t) + \frac{\partial}{\partial t}P_n^{1}(x,t) + (\lambda + \mu_1(x) + \alpha)P_n^{1}(x,t) = \lambda \sum_{k=1}^{\infty} C_k P_{n-k}^{1}(x,t) \,, n \geq 1 \quad (3.1)$$

$$\frac{\partial}{\partial x}P_0^{1}(x,t) + \frac{\partial}{\partial t}P_n^{1}(x,t) + (\lambda + \mu_1(x) + \alpha)P_0^{1}(x,t) = 0 \quad (3.2)$$

$$\frac{\partial}{\partial x}P_n^{2}(x,t) + \frac{\partial}{\partial t}P_n^{2}(x,t) + (\lambda + \mu_2(x) + \alpha)P_n^{2}(x,t) = \lambda \sum_{k=1}^{\infty} C_k P_{n-k}^{2}(x,t) \,, n \geq 1 \quad (3.3)$$

$$\frac{\partial}{\partial x}P_0^{2}(x,t) + \frac{\partial}{\partial t}P_n^{2}(x,t) + (\lambda + \mu_2(x) + \alpha)P_0^{2}(x,t) = 0 \quad (3.4)$$

$$\frac{d}{dt}V_n(t) + (\lambda + \gamma)V_n(t) = \lambda \sum_{k=1}^{\infty} C_k V_{n-k}(t) \,, \quad n \geq 1 \quad (3.5)$$

$$\frac{d}{dt}V_0(t) + (\lambda + \gamma)V_0(t) = \gamma V_0(t) + (1-r)\left[\int_0^{\infty} P_0^1(x,t)\mu_1(x)\,d + \int_0^{\infty} P_0^2(x,t)\mu_2(x)\,dx\right] \quad (3.6)$$

$$\frac{d}{dt}R_0(t) + (\lambda + \beta)R_0(t) = 0 \quad (3.7)$$

$$\frac{d}{dt}R_n(t) + (\lambda + \beta)R_n(t) = \lambda \sum_{k=1}^{\infty} C_k R_{n-k}(t) + \alpha \int_0^{\infty} P_{n-1}^{1}(x,t) + \alpha \int_0^{\infty} P_{n-1}^{2}(x,t) \,, n \geq 1 \quad (3.8)$$

Equations are to be solved subject to the following boundary conditions:

$$P_0^{(1)}(0,t) = \gamma V_1(t) + \beta R_1(t) + (1-r)\left[\int_0^{\infty} P_1^{1}(x,t)\mu_2(x)\,dx + \int_0^{\infty} P_1^{2}(x,t)\mu_2(x)\,dx\right] \quad (3.9)$$

$$P_n^{(1)}(0) = \gamma V_{n+1}(t) + \beta R_{n+1}(t) + (1-r)\int_0^\infty P_{n+1}^{(1)}(x,t)\,\mu_2(x)\,dx + \int_0^\infty P_{n+1}^{(2)}(x,t)\,\mu_2(x)\,dx, \quad n \geq 1 \tag{3.10}$$

$$P_n^{(2)}(0) = \int_0^\infty P_n^{(1)}(x)\,\mu_1(x)\,dx, \quad n \geq 0 \tag{3.11}$$

4. TIME DEPENDENT SOLUTION

Generating functions of the queue length

Now we define the probability generating function as follows

$$P^{(1)}(x,t) = \sum_0^\infty P_n^{(1)}(x,z,t)z^n \quad ; \quad P^{(1)}(z,t) = \sum_0^\infty P_n^{(1)}(t)z^n, |z| \leq 1, x > 0$$

$$P^{(2)}(x,z,t) = \sum_0^\infty P_n^{(2)}(x,t)z^n \quad ; \quad P^{(2)}(z,t) = \sum_0^\infty P_n^{(2)}(t)z^n, |z| \leq 1, x > 0$$

$$V(z,t) = \sum_0^\infty z^n V_n(t) \quad ; \quad R(z,t) = \sum_0^\infty z^n R_n(t) \quad ; \quad C(z) = \sum_0^\infty C_n z^n, \ |z| \leq 1 \tag{4.1}$$

Taking Laplace transforms of equations (3.1) to (3.11)

$$\frac{\partial}{\partial x} P_n^{(1)}(x,s) + (s+\lambda+\mu_1(x)+\alpha)P_n^{(1)}(x,s) = \lambda \sum_{k=1}^\infty C_k P_{n-k}^{(1)}(x,s), \quad n \geq 1 \tag{4.2}$$

$$\frac{\partial}{\partial x} P_0^{(1)}(x,s) + (s+\lambda+\mu_1(x)+\alpha)P_0^{(1)}(x,s) = 0 \tag{4.3}$$

$$\frac{\partial}{\partial x} P_n^{(2)}(x,s) + (s+\lambda+\mu_1(x)+\alpha)P_n^{(2)}(x,s) = \lambda \sum_{k=1}^\infty C_k P_{n-k}^{(2)}(x,s), \quad n \geq 1 \tag{4.4}$$

$$\frac{\partial}{\partial x} P_0^{(2)}(x,s) + (s+\lambda+\mu_1(x)+\alpha)P_0^{(2)}(x,s) = 0 \tag{4.5}$$

$$(s+\lambda+\gamma)V_0(s) = 1 + (1-r)\int_0^\infty P_0^{(1)}(x,s)\,\mu_2(x)\,dx + \int_0^\infty P_0^{(2)}(x,s)\,\mu_2(x)\,dx + \gamma V_0(s) \tag{4.6}$$

$$(s+\lambda+\gamma)V_n(s) = \lambda V_{n-1}(s), \quad n \geq 1 \tag{4.7}$$

$$(s+\lambda+\beta)R_0(s) = \alpha\int_0^\infty P_0^{(1)}(x,s)\,\mu_1(x)\,dx + \alpha\int_0^\infty P_0^{(2)}(x,s)\,\mu_2(x)\,dx \tag{4.8}$$

$$(s+\lambda+\beta)R_n(s) = \lambda R_{n-1}(s), \quad n \geq 1 \tag{4.9}$$

$$P_0^{(1)}(o,s) = \gamma V_1(s) + \beta R_1(s) + (1-r)\int_0^\infty P_1^{(1)}(x,s)\,\mu_2(x)\,dx + q\int_0^\infty P_1^{(2)}(x,s)\,\mu_2(x)\,dx \tag{4.10}$$

$$P_n^{1}(o,s) = \gamma V_1(s) + \beta R_1(s) + (1-r)\int_0^\infty P_{n+1}^{1}(x,s)\mu_1(x)dx + \int_0^\infty P_{n+1}^{2}(x,s)\mu_2(x)dx, \quad n \geq 1 \tag{4.11}$$

$$P_n^{2}(o,s) = \int_0^\infty P_n^{(1)}(x,s)\mu_1(x)dx, n \geq 0 \tag{4.12}$$

We multiply both sides of equations (4.2) and (4.3) by suitable powers of z, sum over n and use (4.1) and simplify. We thus have after algebraic simplifications

$$\frac{\partial}{\partial x}P^{(1)}(x,z,s) + [s+\lambda-\lambda C(z)+\mu_1(x)+\alpha]P^{(1)}(x,z,s) = 0 \tag{4.13}$$

Performing similar operations on equations (4.4) and (4.5) and using (4.1), We have

$$\frac{\partial}{\partial x}P^{2}(x,z,s) + (s+\lambda-\lambda C(z)+\mu_1(x)+\alpha)P^{2}(x,z,s) = 0 \tag{4.14}$$

Similar operations on equations (4.6),(4.7),(4.8) and (4.9) yields

$$(s+\lambda-\lambda C(z)+\gamma)V(z,s) = 1 + (1-r)\int_0^\infty P_0^{1}(x,s)\mu_2(x)dx + \int_0^\infty P_0^{2}(x,s)\mu_2(x)dx + \gamma V_0(s) \tag{4.15}$$

$$(s+\lambda-\lambda C(z)+\beta)R(z,s) = \alpha z\int_0^\infty P^{(1)}(x,z,s)dx + \alpha z\int_0^\infty P^{2}(x,z,s)dx \tag{4.16}$$

Now We multiply both sides of equation (4.10) by z, multiply both sides of equation (4.11) by z^{n+1}, sum over n from 1to ∞, add the two results and use (4.1)&(4.6).Thus we obtain after mathematical adjustments

$$z P^{(1)}(0,z,s) = (1-r)\int_0^\infty P^{1}(x,z,s)\mu_1(x)dx + \int_0^\infty P^{2}(x,z,s)\mu_2(x)dx + \gamma V(z,s) - (1-r)\int_0^\infty P_0^{1}(x,s)\mu_1(x)dx - \int_0^\infty P_0^{2}(x,s)\mu_2(x)dx + \beta R(z,s) \tag{4.17}$$

$$P^{2}(0,z,s) = \int_0^\infty P^{(1)}(x,z,s)\mu_1(x)dx \tag{4.18}$$

Using (4.15 in (4.17), we get

$$z P^{(1)}(0,z,s) = (1-r)\int_0^\infty P^{1}(x,z,s)\mu_1(x)dx + \int_0^\infty P^{2}(x,z,s)\mu_2(x)dx + 1 - [s+\lambda-\lambda C(z)]V(z,s) + \beta R(z,s) \tag{4.19}$$

Integrating equations (4.2), (4.3) and (4.4) between 0 and x, we get

$$P^{(1)}(x,z,s) = P^{(1)}(0,z,s)\, e^{-(s+\lambda-\lambda C(z)+\alpha)x-\int_0^{x}\mu_1(t)dt} \quad (4.20)$$

$$P^{(2)}(x,z,s) = P^{(2)}(0,z,s)\, e^{-(s+\lambda-\lambda C(z)+\alpha)x-\int_0^{x}\mu_2(t)dt} \quad (4.21)$$

Again integrating equation (4.10) w.r.to x, we have

$$P^{(1)}(z,s) = P^{(1)}(0,z,s)\left[\frac{1-B_1(s+\lambda-\lambda C(z)+\alpha)}{s+\lambda-\lambda C(z)+\alpha}\right] \quad (4.22)$$

$$where\ B_1(s+\lambda-\lambda C(z)+\alpha) = \int_0^{\infty} e^{-(s+\lambda-\lambda C(z)+\alpha)x} dB_1(x) \quad (4.23)$$

is the Laplace transform of first stage of service time.

Now from equation (4.10) after some simplification and using equation (1.1) , we obtain

$$\int_0^{\infty} P^{(1)}(x,z,s)\mu_1(x)dx = P^{(1)}(0,z,s)B_1(s+\lambda-\lambda C(z)+\alpha) \quad (4.24)$$

Again integrating equation (4.11) w.r.to x, we have

$$P^{(2)}(z,s) = P^{(2)}(0,z,s)\left[\frac{1-B_2(s+\lambda-\lambda C(z)+\alpha)}{s+\lambda-\lambda C(z)+\alpha}\right] \quad (4.25)$$

$$where\ B_2(s+\lambda-\lambda C(z)+\alpha) = \int_0^{\infty} e^{-(s+\lambda-\lambda C(z)+\alpha)x} dB_2(x) \quad (4.26)$$

is the Laplace transform of second stage of service time.

Now from equation (4.11) after some simplification and using equation (1.1) , we obtain

$$\int_0^{\infty} P^{(2)}(x,z,s)\mu_2(x)dx = P^{(2)}(0,z,s)B_2(s+\lambda-\lambda C(z)+\alpha) \quad (4.27)$$

Using (4.24)& (4.27) in (4.16) we get,

$$(s+\lambda-\lambda C(z)+\beta)R(z) = \alpha z P^{(1)}(0,z,s)\left(\frac{[1-B_1(s+\lambda-\lambda C(z)+\alpha)B_2(s+\lambda-\lambda C(z)+\alpha)]}{s+\lambda-\lambda C(z)+\alpha}\right) \quad (4.28)$$

Now using equations (4.18) (4.21), (4.23),(4.24),(4.26) and (4.27) in equation (4.19) and solving for $P^{(1)}(0,z)$ we get

$$P^{(1)}(0,z,s) = \frac{f_1(z)f_2(z)[1-(s+\lambda-\lambda C(z))V(z,s)]}{DR} \quad (4.29)$$

Where $DR = f_1 \ z \ f_2 \ z \quad z - \ 1 - r \ B_1 \ s + \lambda - \lambda C \ z \ + \alpha \ - r \, B_1 \ s + \lambda - \lambda C \ z \ + \alpha \ B_2 \ s + \lambda - \lambda C \ z \ + \alpha \quad - \alpha\beta z(1 - \ 1 - r \ B_1 \ s + \lambda - \lambda C \ z \ + \alpha \ - r \, B_1 \ s + \lambda - \lambda C \ z \ + \alpha \ B_2 \ s + \lambda - \lambda C \ z \ + \alpha$

(4.30)

$$f_1 \ \mathrm{z} \ = \mathrm{s} + \lambda - \lambda \mathrm{C} \ \mathrm{z} \ + \ \alpha \ \text{ and } f_2 \ \mathrm{z} \ = \mathrm{s} + \lambda - \lambda \mathrm{C}(\mathrm{z}) \ + \ \beta$$

Substituting the value of $P^{\ 1} \ \ 0, z$ from equation (4.22) into equations (4.13), (4.16) & (4.18) we get

$$P^{(1)} \ z, s \ = \frac{f_2 \ z \ [1 - B_1 \ s + \lambda - \lambda C \ z \ + \alpha \ ]}{DR} [1 - \ \ s + \lambda - \lambda C \ z \ \ \ V \ z, s \ ] \qquad (4.31)$$

$$P^{\ 2} \ \ z, s \ = \frac{f_2 \ z \ B_1 \ s + \lambda - \lambda C \ z \ + \alpha \ [1 - B_2 \ s + \lambda - \lambda C \ z \ + \alpha \ ]}{DR} [1 - \ \ s + \lambda - \lambda C \ z \ \ \ V \ z, s \ ] \qquad (4.32)$$

$$R \ \ z, s \ \ = \frac{\alpha z \ 1 - \ 1 - r \ B_1 \ s + \lambda - \lambda C \ z \ + \alpha \ - r \, B_1 \ s + \lambda - \lambda C \ z \ + \alpha \ B_2 \ s + \lambda - \lambda C \ z \ + \alpha}{DR} [1 - \ \ s + \lambda - \lambda C \ z \ \ \ V \ z, s \ ] \qquad (4.33)$$

In this section we shall derive the steady state probability distribution for our Queuing model. To define the steady state probabilities, suppress the arguments where ever it appears in the time dependent analysis. By using well known Tauberian property,

$$\lim_{s \to 0} s f \ \ s \ \ = \lim_{t \to \infty} f(t)$$

$$P^{\ 1} \ \ z \ \ = \frac{f_2 \ z \ \ 1 - B_1 \ \lambda - \lambda C \ z \ + \alpha}{DR} \lambda(C \ \ z \ \ - 1) V \ \ z \qquad (4.34)$$

$$P^{\ 2} \ \ z \ \ = \frac{f_2 \ z \ B_1 \ \lambda - \lambda C \ z \ + \alpha \ \ 1 - B_2 \ \lambda - \lambda C \ z \ + \alpha}{DR} \lambda(C \ \ z \ \ - 1) V \ \ z \qquad (4.35)$$

$$R \ \ z \ \ = \frac{\alpha z [1 - \ 1 - r \ B_1 \ s + \lambda - \lambda C \ z \ + \alpha \ - r \, B_1 \ s + \lambda - \lambda C \ z \ + \alpha \ B_2 \ s + \lambda - \lambda C \ z \ + \alpha \ ]}{DR} \lambda(C \ \ z \ \ - 1) V \ \ z \qquad (4.36)$$

In order to determine $P^{\ 1} \ \ z \ , P^{\ 2} \ \ z \ , \mathrm{R(z)}$ completely, we have yet to determine the unknown $V(1)$ which appears in the numerator of the right sides of equations (4.34), (4.35) and (4.36). For that purpose, we shall use the normalizing condition.

$$P^{\ 1} \ \ 1 \ + P^{\ 2} \ \ 1 \ + V \ \ 1 \ + R \ \ 1 \ \ = 1 \qquad (4.37)$$

$$P^{\ 1} \ (1) = \frac{\lambda \beta C^{'} \ 1 \ \ \ 1 - B_1 \ \alpha}{dr} V(1) \qquad (4.38)$$

$$P^{2}(1) = \frac{\lambda\beta C'(1)\, B_1(\alpha)(1-B_2(\alpha))}{dr} V(1) \tag{4.39}$$

$$R(1) = \frac{\lambda\alpha C'(1)\,(1-(1-r)B_1(\alpha) - rB_1(\alpha) B_2(\alpha))}{dr} V(1) \tag{4.40}$$

Where $dr = \alpha\beta(1-p)B_1(\alpha) B_2(\alpha) - [1 - B_1(\alpha) B_2(\alpha)]\,\square\lambda C'(1)(\alpha+\beta)$

$P^{1}(1), P^{2}(1)$ *and* $R(1)$ denote the steady state probabilities that the server is providing first stage of service, second stage of service and server under repair without regard to the number of customers in the queue. Now using equations (4.38), (4.39), (4.40) into the normalizing condition (4.37) and simplifying, we obtain

$$V(1) = 1 - \frac{\square\lambda C'(1)}{\beta[(1-r)B_1(\alpha) + rB_1(\alpha) B_2(\alpha)]} - \frac{\square\lambda C'(1)}{\alpha[(1-r)B_1(\alpha) + rB_1(\alpha) B_2(\alpha)]} + \frac{\square\lambda C'(1)}{\beta} + \frac{\square\lambda C'(1)}{\alpha} \tag{4.41}$$

and hence, the utilization factor ρ of the system is given by

$$\rho = \frac{\square\lambda C'(1)}{\beta[(1-r)B_1(\alpha) + rB_1(\alpha) B_2(\alpha)]} + \frac{\square\lambda C'(1)}{\alpha[(1-r)B_1(\alpha) + rB_1(\alpha) B_2(\alpha)]} - \frac{\square\lambda C'(1)}{\beta} - \frac{\square\lambda C'(1)}{\alpha} \tag{4.42}$$

Where $\rho < 1$ is the stability condition under which the steady states exits.

5. The Mean queue size and the mean system size

Let $P_q(\mathrm{z})$ denote the probability generating function of the queue size irrespective of the server state. Then adding equation (4.27), (4.28) and (4.29) we obtain

$$P_q(\mathrm{z}) = P^{1}(z) + P^{2}(z) + R(z)$$

$$P_q(\mathrm{z}) = \frac{\mathrm{N(z)}}{\mathrm{D(z)}} \tag{5.1}$$

$$\begin{aligned} N(z) = {} & (\lambda C(z) - 1)[1-(1-r)B_1(\lambda - \lambda C(z) + \alpha) \\ & - rB_1(\lambda - \lambda C(z) + \alpha) B_2(\lambda - \lambda C(z) + \alpha)](\alpha z + f_2(z))V(z) \end{aligned}$$

$$\begin{aligned} D(z) = {} & f_1(z) f_2(z)[z-(1-r)B_1(\lambda - \lambda C(z) + \alpha) - rB_1(\lambda - \lambda C(z) + \alpha) B_2(\lambda - \lambda C(z) + \alpha)] \\ & -\alpha\beta z[1-(1-r)B_1(\lambda - \lambda C(z) + \alpha) - rB_1(\lambda - \lambda C(z) + \alpha) B_2(\lambda - \lambda C(z) + \alpha)] \end{aligned}$$

Let L_q denote the mean number of customers in the queue under the steady state. Then we have

$$L_q = \frac{d}{dz}\,[P_q\ z\ ] \text{ at } z = 1$$

$$L_q = \lim_{z\to 1} \frac{D'\ 1\ N''\ 1\ - N'\ 1\ D''(1)}{2D'(1)^2} \tag{5.2}$$

where primes and double primes in (4.36) denote first and second derivative at z = 1, respectively. Carrying out the derivative at z = 1 we have

$$N'\ 1\ = \lambda\, C'\ 1\ \ \alpha + \beta\ V(1)(1-(1-\mathrm{r})B_1\ \alpha\ - rB_1\ \alpha\ B_2\ \alpha\ ) \tag{5.3}$$

$$\begin{aligned} N''\ 1\ = {} & (1-(1-\mathrm{r})B_1\ \alpha\ - rB_1\ \alpha\ B_2\ \alpha\ )\{\lambda C''\ 1\ \ \alpha+\beta\ V\ 1\ - 2\ \lambda C'\ 1\ ^2 V\ 1 \\ & + 2\lambda C'\ 1\ \alpha V\ 1\ + 2\lambda C'\ 1\ \ \alpha+\beta\ V'\ 1\ \} \\ & - 2\lambda^3\ C'\ 1\ ^3\ \alpha+\beta\ V(1)[\ 1-r\ B_2'\ \alpha\ + r\, B_1\ \alpha\ B_2'\ \alpha\ + rB_2\ \alpha\ B_1'\ \alpha\ ] \end{aligned} \tag{5.4}$$

$$\begin{aligned} D'\ 1\ = {} & \alpha\beta\ 1-r\ B_1'\ \alpha\ + r\, B_1\ \alpha\ B_2'\ \alpha \\ & -\ \ 1-r\ B_1'\ \alpha\ + r\, B_1\ \alpha\ B_2'\ \alpha\ \ [\ \alpha+\beta\ \lambda C'\ 1\ ]'' \end{aligned} \tag{5.5}$$

$$\begin{aligned} D''\ 1\ = {} & 2\alpha\beta(1-\mathrm{r})[B_1'\ \alpha\ - r(B_1\ \alpha\ B_2'(\alpha) + B_2\ \alpha\ B_2'(\alpha))] -\ \alpha+\beta\ \lambda C''\ 1\ (1-(1 \\ & -\ \mathrm{r})B_1\ \alpha\ - rB_1\ \alpha\ B_2\ \alpha\ - 2\ \alpha+\beta\ \lambda C'\ 1\ [1-(1-\mathrm{r})B_1'\ \alpha \\ & - r(B_1\ \alpha\ B_2'\ \alpha\ +\ B_2\ \alpha\ B_2'(\alpha) \end{aligned} \tag{5.6}$$

Then if we substitute the values from (5.3), (5.4), (5.5) and (5.6) into (5.2) we obtain L_q in the closed form. Further we find the mean system size L using Little's formula. Thus we have

$$L = L_q + \rho$$

where L_q has been found by equation (5.2) and ρ is obtained from equation (4.35).

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