$M^{[X]}$/G/1 Feedback Queue with Two-Stage Heterogeneous Service Multiple Vacation and Random Breakdowns

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Abstract: This paper studies a batch arrival single server Bernoulli feedback queue with Poisson arrivals, two stages of heterogeneous service with different (arbitrary) service time distributions subject to random breakdowns and multiple vacations with exponential distributed with Mean vacation time $\frac{1}{\gamma}$. After first-stage service the server must provide the second stage service. However after the completion of two stage of service, if the customer is dissatisfied with his service or if he received unsuccessful service, then the customer may immediately join the tail of the original queue with probability p $(0 \leq p < 1)$ to receive first stage of service. Otherwise the customer may depart forever from the system with probability $q = (1-p)$. The server takes vacation each time when the system becomes empty. On returning from vacation if the server finds no customer waiting in the system, then the server again goes for vacation until he finds at least one customer in the system. The system may breakdown at random and repair time follow exponential distribution. The time dependent probability generating functions has been obtained in terms of their Laplace transforms and the corresponding steady state results have been obtained explicitly. Also the average number of customers in the queue and the average waiting time are derived. The probability generating function of queue size at a random epoch is obtained. Particular cases and some special cases are discussed. Numerical illustration for particular values of parameters is presented.

Keywords: Batch arrival, Probability Generating function, Random breakdown, multiple vacations.

1. Introduction

Batch queueing models have been analyzed in the past by several authors. Server vacation models are useful for the systems in which the server wants to utilize the idle time for different purposes. Application of vacation models can be found in production line systems, designing local area networks and data communication systems. Queueing models with vacations have been investigated by many authors including Keilson and Servi [14], Cramer [23], Scholl and kleinrock [24], Shanthakumar [16], Doshi [5] and [6], Madan [18], [19], [20] .Choudhuryand Madan [9] have studied a queueing system with Bernoulli schedule server vacation. Chae et al. [17], Chang and Takine [30] and Igaki [25] have studied queues with generalized vacations. Vacation queue with c servers has beenstudied by Tian et al. [27]. Choudhury and Borthakur [8] and Hur and Ahn [29] have studied vacation ueues with batch arrivals. Queue with multiple vacations has been studied by Tian and Zhang [26].

Recently there have been several contributions considering queueing systems of *M/G/*1 type in which the server may provide a second phase of service. One may refer to Bertsimas and Papaconstantinou [6], Madan [19], [20], Choudhury [9], [10] and [11], Medhi [13],Kalyanaraman[26], Krishna Kumar [2], Badamchi and Shahkar[1] have also studied a single server queue with two phase queueing system with Bernoulli feedback and Bernoulli schedule server vacation.

Takine and Sengupta [31], Aissani and Artalejo [1] have studied different queueing systems subject to random breakdowns. Kulkarni and Choi [13] and Wang et al. [33] have studied retrial queues with system breakdowns and repairs. Thangaraj and Vanitha [32] discussed a M/G/1 queue with two stage heterogeneous service compulsory server vacation and random breakdowns.

In this paper, we analyze a single server Bernoulli feedback queue with two stages of service subject to random breakdowns having multiple server vacation, where the customers arrive to the system in batches of variable size. However after the completion of two stage of service, if the customer is dissatisfied with his service for certain reason or if he received unsuccessful service, then the customer may immediately join the tail of the original queue with probability p $(0 \leq p < 1)$. Otherwise the customer may depart forever from the system with probability $q = (1-p)$. If there is no customer waiting in the system then the server goes for vacation with random duration. It follows general distribution. On returning from vacation, if the server again founds no customer waiting in the system, then he goes for vacation. The server continues to go for vacation until he finds at least one customer in the system. And once the system break down, it enters a repair process and the customer whose service is interrupted goes back to the head of the queue where the arrivals are Poisson. The customers arrive to the system one by one and are served on a first come-first served basis. Particular cases and some special cases are discussed. Numerical illustration for particular values of parameters is presented.

2. Mathematical Description of the model

We assume the following to describe the queueing model of our study.

- Customers arrive at the system in batches of variable size in a compound Poisson process and they are provided service one by one on a first come first served basis. Let $\lambda \Sigma C_k\, dt$ $(k = 1, 2, 3, \dots)$ be the first order probability that a batch of k customers arrives at the system during a short interval of time (t, t + dt], $where\ 0 \leq C_k \leq 1\ and\ \sum_{k=1}^{\infty} C_k = 1\ and\ \lambda > 0$ is the mean arrival rate of batches.

- Each customer undergoes two stages of heterogeneous service provided by a single server on a first come first served basis. The service time of the two stages follow different general (arbitrary) distributions with distribution function $B_j(t)$ and the density function $b_j(t)$, $j = 1, 2$

- As soon as the second stage of a customer is completed and if the customer is dissatisfied with his service for certain reason or if he received unsuccessful service, the customer may immediately join the tail of the original queue with probability p $(0 \leq p < 1)$ Otherwise the customer may depart forever from the system with probability $q = (1 - p)$

- Let $\mu_j(x)dx$ be the conditional probability of completion of the jth stage of service during the interval $(x, x + dx]$ given that elapsed time is x, so that

$$\mu_j(x) = \frac{bj(x)}{1 - B_j(x)};\ j = 1,2$$

and therefore,

$$bj(t) = \mu_j(t) e^{-\left(\int \mu_j(x)\,dx\right)}; \quad j = 1, 2 \tag{2.1}$$

- The customers are served according to the first come, first served rule.

- If there is no customer waiting in the queue, then the server goes for a vacation. The vacation periods are exponentially distributed with mean vacation time $\frac{1}{\gamma}$. On returning from vacation if the server again founds no customer in the queue, then it goes for another vacation. So the server takes multiple vacations

- The customer both newly arrived and those that are feedback are served in the order in which they join the tail of the original queue. Also service time for a feedback customer is independent of its previous service time.

- The system may break down at random and breakdowns are assumed to occur according to a Poisson stream with mean breakdown rate $\alpha > 0$. Further we assume that once the system breaks down, the customer whose service is interrupted comes back to the head of the queue.

- Once the system breaks down, it enters a repair process immediately. The repair times are exponentially distributed with mean repair rate $\beta > 0$.

- Various stochastic processes involved in the system are assumed to be independent of each other.

3. Definitions and Equations Governing the System

Let $NQ(t)$ denote the queue size(excluding one in service) at time t. We introduce the random variable Y(t) as follows

$$Y(t) = \begin{cases} 1, \textit{if the system is busy with first stage of service at time } t \\ 2, \textit{if the system is busy with second stage of service at time } t \\ 3, \textit{if the system is on vacation at time } t \\ 4, \textit{if the system is inactive due to breakdown at time } t \end{cases}$$

We introduce the supplementary variable as,

$$L(t) = \begin{cases} B_1^0(t) & if \ Y(t) = 1 \\ B_2^0(t) & if \ Y(t) = 2 \\ V^0(t) & if \ Y(t) = 3 \\ R^0(t) & if \ Y(t) = 4 \end{cases}$$

where

$B_1^0(t)$ = elapsed service time for the first stage of service at time t,
$B_2^0(t)$ = elapsed service time for the second stage of service at time t,
$V^0(t)$ = elapsed vacation time of the server at time t.
$R^0(t)$ = elapsed repair time of the server at time t.

The process $\{NQ(t)$, L(t) $\}$ is a continuous time Markov process. We define the probabilities for i = 1,2.$P_n^i(x,t) = Prob\{NQ(t) = n, L(t) = B_i^0 ; x < B_i^0 \le x + dx\} ; x > 0, n > 0$

$Vn(x, t) = Prob\ \{NQ(t) = n, L(t) = V0(t)\ ; x < V0(t) \leq x + dx\}, x > 0, n \geq 0$
In steady state condition, we have

$P_n^i(x)\,dx = \lim_{t\to\infty} P_n^i\ (x, t),\quad i=1, 2\ \ x > 0\ ;\ n \geq 0$

$V_n = \lim_{t\to\infty} V_n\ t \quad ; \quad n \geq 0$

$R_n = \lim_{t\to\infty} R_n\ t\ ;\ n \geq 0$

Assume that

$V_0\ 0\ = 1, V_n\ 0\ = 0$

And for i=1,2

$B_i\ 0\ , B_i\ \infty\ = 1$
Also V(x) and $B_i(x)$ are continuous at x =0.

We define
$P_n^{(1)}\ x, t\ = \Pr$ {at time t, the server is active providing first stage of service and there are n ($n \geq 0$) customers in the queue excluding the one customer in the first stage of service being served and the elapsed service time for this customer is x}. $P_n^{(1)}(t) = \int_0^\infty P_n^{(1)}\ x, t\ dx$ denote the probability that at time t there are n customers in the queue excluding one customer in the first stage of service irrespective of the value of x.

$P_n^{(2)}\ x, t\ = \Pr$ {at time t, the server is active providing second stage of service and there are n ($n \geq 0$) customers in the queue excluding the one customer in the second stage of service being served and the elapsed service time for this customer is x}. $P_n^{(2)}(t) = \int_0^\infty P_n^{(2)}\ x, t\ dx$ denotes the probability that at time t there are n customers in the queue excluding one customer in the second stage of service irrespective of the value of x.

$V_n(t)$ =Pr {at time t, the server is on vacation, while there are n ($n \geq 0$) customers in the queue}.

$R_n(t)$ =Pr {at time t, the server is inactive due to system breakdown and the system is under repair, while there are n ($n \geq 0$) customers in the queue}.

The model is then, governed by the following set of differential-difference equations:

$$\frac{\partial}{\partial x} P_n^{\ 1}\ x, t\ + \frac{\partial}{\partial t} P_n^{\ 1}\ x, t\ + (\lambda + \mu_1(x) + \alpha) P_n^{\ 1}\ (x, t)\ = \lambda \sum_{k=1}^{\infty} C_k P_{n-k}^{\ 1}\ x, t\ ,\ n \geq 1 \quad (3.1)$$

$$\frac{\partial}{\partial x} P_0^{\ 1}\ x, t\ + \frac{\partial}{\partial t} P_n^{\ 1}\ x, t\ + \ \lambda + \mu_1(x) + \alpha\ P_0^{\ 1}\ x, t\ = 0 \quad (3.2)$$

$$\frac{\partial}{\partial x} P_n^{\ 2}\ x, t\ + \frac{\partial}{\partial t} P_n^{\ 2}\ x, t\ + (\lambda + \mu_2(x) + \alpha) P_n^{\ 2}\ (x, t)\ = \lambda \sum_{k=1}^{\infty} C_k P_{n-k}^{\ 2}\ x, t\ ,\ n \geq 1 \quad (3.3)$$

$$\frac{\partial}{\partial x} P_0^{\ 2}\ x, t\ + \frac{\partial}{\partial t} P_n^{\ 2}\ x, t\ + \ \lambda + \mu_2(x) + \alpha\ P_0^{\ 2}\ x, t\ = 0 \quad (3.4)$$

$$\frac{d}{dt}V_n(t) + (\lambda+\gamma)V_n(t) = \lambda \sum_{k=1}^{\infty} C_k V_{n-k}(t), \quad n \geq 1 \tag{3.5}$$

$$\frac{d}{dt}V_0(t) + (\lambda+\gamma)V_0(t) = \gamma V_0(t) + q\int_0^{\infty} P_0^2(x,t)\mu_2(x)dx \tag{3.6}$$

$$\frac{d}{dt}R_0(t) + (\lambda+\beta)R_0(t) = 0 \tag{3.7}$$

$$\frac{d}{dt}R_n(t) + (\lambda+\beta)R_n(t) = \lambda \sum_{k=1}^{\infty} C_k R_{n-k}(t) + \alpha\int_0^{\infty} P_{n-1}^{1}(x,t)\,d(x) + \alpha\int_0^{\infty} P_{n-1}^{2}(x,t)\,d(x), \; n \geq 1 \tag{3.8}$$

Equations are to be solved subject to the following boundary conditions:

$$P_0^{(1)}(0,t) = \gamma V_1(t) + \beta R_1(t) + p\int_0^{\infty} P_0^{(2)}(x,t)\mu_2(x)dx + q\int_0^{\infty} P_1^{(2)}(x,t)\mu_2(x)dx \tag{3.9}$$

$$P_n^{(1)}(0,t) = \gamma V_{n+1}(t) + \beta R_{n+1}(t) + p\int_0^{\infty} P_n^{(2)}(x,t)\mu_2(x)dx + q\int_0^{\infty} P_{n+1}^{(2)}(x,t)\mu_2(x)dx, \; n \geq 1 \tag{3.10}$$

$$P_n^{(2)}(0,t) = \int_0^{\infty} P_n^{(1)}(x,t)\mu_1(x)dx, \; n \geq 0 \tag{3.11}$$

4. Steady State Probability Generating Functions of the Queue Size

In this section we obtain the transient solution for the above set of differential difference equation.

We define the probability generating functions,

$$P^{(1)}(x,t) = \sum_0^{\infty} P_n^{(1)}(x,z,t)z^n \quad ; \quad P^{(1)}(z,t) = \sum_0^{\infty} P_n^{(1)}(t)z^n, |z| \leq 1, x > 0$$

$$P^{(2)}(x,z,t) = \sum_0^{\infty} P_n^{(2)}(x,t)z^n \quad ; \quad P^{(2)}(z,t) = \sum_0^{\infty} P_n^{(2)}(t)z^n, |z| \leq 1, x > 0$$

$$V(z,t) = \sum_0^{\infty} z^n V_n(t) \quad ; \quad R(z,t) = \sum_0^{\infty} z^n R_n(t) \quad ; \quad C(z) = \sum_0^{\infty} C_n z^n, \; |z| \leq 1 \tag{4.1}$$

Taking Laplace transforms of equations (3.1) to (3.11)

$$\frac{\partial}{\partial x}\bar{P}_n^{(1)}(x,s) + (s+\lambda+\mu_1(x)+\alpha)\bar{P}_n^{(1)}(x,s) = \lambda \sum_{k=1}^{\infty} C_k \bar{P}_{n-k}^{(1)}(x,s), \quad n \geq 1 \tag{4.2}$$

$$\frac{\partial}{\partial x}\bar{P}_0^{(1)}(x,s) + (s+\lambda+\mu_1(x)+\alpha)\bar{P}_0^{(1)}(x,s) = 0 \tag{4.3}$$

$$\frac{\partial}{\partial x}\bar{P}_n^{(2)}(x,s) + (s+\lambda+\mu_1(x)+\alpha)\bar{P}_n^{(2)}(x,s) = \lambda \sum_{k=1}^{\infty} C_k \bar{P}_{n-k}^{(2)}(x,s), \quad n \geq 1 \tag{4.4}$$

$$\frac{\partial}{\partial x}\bar{P}_0^{(2)}(x,s) + (s+\lambda+\mu_1(x)+\alpha)\bar{P}_0^{(2)}(x,s) = 0 \tag{4.5}$$

$$s+\lambda+\gamma\ V_0\ s\ =1+q\ \int_0^\infty P_0^{\ 2}\ \ x,s\ \mu_2\ x\ dx+\gamma V_0\ s \tag{4.6}$$

$$s+\lambda+\gamma\ V_n\ s\ =\lambda\ \sum_{k=1}^\infty C_k V_{n-k}\ \ s\ ,\ \ n\geq 1 \tag{4.7}$$

$$s+\lambda+\ \beta\ R_0\ s\ =\quad 0 \tag{4.8}$$

$$s+\lambda+\beta\ R_n\ s\ =\lambda\sum_{k=1}^{\infty} C_k\, R_{n-k}\ \ s\ \alpha\int_0^\infty P_{n-1}^{\ 1}\ x,s\ dx+\alpha\int_0^\infty P_{n-1}^{\ 2}(x,s)dx,\quad n\geq 1 \tag{4.9}$$

$$P_0^{\ 1}\ \ o,s\ =\gamma V_n\ s\ +\beta R_n\ s\ +p\int_0^\infty P_0^{\ 2}\ \ x,s\ \mu_2\ x\ dx+q\int_0^\infty P_1^{\ 2}\ \ x,s\ \mu_2\ x\ dx \tag{4.10}$$

$$P_n^{\ 1}\ \ o,s\ =\gamma V_n\ s\ +\beta R_n\ s\ +p\int_0^\infty P_n^{\ 2}\ \ x,s\ \mu_2\ x\ dx+q\int_0^\infty P_{n+1}^{\ 2}\ x,s\ \mu_2\ x\ dx, n\geq 1 \tag{4.11}$$

$$P_n^{\ 2}\ \ o,s\ =\int_0^\infty P_0^{(1)}\ x,s\ \mu_1\ x\ dx\ , n\geq 0 \tag{4.12}$$

We multiply both sides of equations (4.2) and (4.3) by suitable powers of z, sum over n and use (4.1) and simplify. We thus have after algebraic simplifications

$$\frac{\partial}{\partial x}P^{(1)}\ x,z,s\ +[s+\lambda-\lambda C(z)+\mu_1(x)+\alpha]P^{(1)}(x,z,s)\ =0 \tag{4.13}$$

Performing similar operations on equations (4.4) and (4.5) and using (4.1),
We have

$$\frac{\partial}{\partial x}P^{\ 2}\ \ x,z,s\ +\ s+\lambda-\lambda C\ z\ +\mu_1\ x\ +\alpha\ P^{\ 2}\ \ x,z,s\ =0 \tag{4.14}$$

Similar operations on equations (4.6),(4.7),(4.8) and (4.9) yields

$$s+\lambda-\lambda C\ z\ +\gamma\ V\ z,s\ =1+q\int_0^\infty P^{\ 2}\ \ x,z,s\ \mu_2\ x\ dx+\gamma V_0\ s \tag{4.15}$$

$$s+\lambda-\lambda C\ z\ +\beta\ R\ z,s\ =\alpha z\int_0^\infty P^{(1)}\ x,z,s\ dx+\alpha z\int_0^\infty P^{\ 2}\ \ x,z,s\ dx \tag{4.16}$$

Now we multiply both sides of equation (4.10) by z, multiply both sides of equation (4.11) by z^{n+1}, sum over n from 1 to ∞, add the two results and use (4.1)&(4.6).Thus we obtain after mathematical adjustments

$$z\,P^{(1)}\ 0,z,s\ =\ \ q+pz\int_0^\infty P^{\ 2}\ \ x,z,s\ \mu_2\ x\ dx+1-[s+\lambda-\lambda C(z]V\ z,s\ +\beta R\ z,s \tag{4.17}$$

$$P^{\ 2}\ (0,z,s)=\int_0^\infty P^{(1)}\ x,z,s\ \mu_1\ x\ dx \tag{4.18}$$

Integrating equations (4.13)and (4.14) between 0 and x, we get

$$P^{(1)}(x,z,s) = P^{(1)}(0,z,s)\, e^{-(s+\lambda-\lambda C(z)+\alpha)x-\int_0^{\infty}\mu_1(t)dt} \qquad (4.19)$$

$$P^{(2)}(x,z,s) = P^{(2)}(0,z,s)\, e^{-(s+\lambda-\lambda C(z)+\alpha)x-\int_0^{\infty}\mu_2(t)dt} \qquad (4.20)$$

Again integrating equation (4.19) w.r.to x, we have

$$P^{(1)}(z,s) = P^{(1)}(0,z,s)\left[\frac{1-B_1(s+\lambda-\lambda C(z)+\alpha)}{s+\lambda-\lambda C(z)+\alpha}\right] \qquad (4.21)$$

$$where\ B_1(s+\lambda-\lambda C(z)+\alpha) = \int_0^{\infty} e^{-(s+\lambda-\lambda C(z)+\alpha)x} dB_1(x) \qquad (4.22)$$

is the Laplace transform of first stage of service time.

Now from equation (4.19) after some simplification and using equation (2.1) , we obtain

$$\int_0^{\infty} P^{(1)}(x,z,s)\mu_1(x)\,dx = P^{(1)}(0,z,s)B_1(s+\lambda-\lambda C(z)+\alpha) \qquad (4.23)$$

Again integrating equation (4.20) w.r.to x, we have

$$P^{(2)}(z,s) = P^{(2)}(0,z,s)\left[\frac{1-B_2(s+\lambda-\lambda C(z)+\alpha)}{s+\lambda-\lambda C(z)+\alpha}\right] \qquad (4.24)$$

$$where\ B_2(s+\lambda-\lambda C(z)+\alpha) = \int_0^{\infty} e^{-(s+\lambda-\lambda C(z)+\alpha)x} dB_2(x) \qquad (4.25)$$

is the Laplace transform of second stage of service time.

Now from equation (4.11) after some simplification and using equation (1.1) , we obtain

$$\int_0^{\infty} P^{(2)}(x,z,s)\mu_2(x)\,dx = P^{(2)}(0,z,s)B_2(s+\lambda-\lambda C(z)+\alpha) \qquad (4.26)$$

Using (4.23)& (4.26) in (4.16) we get,

$$(s+\lambda-\lambda C(z)+\beta)R(z,s) = \alpha z P^{(1)}(0,z,s)\frac{[1-B_1(s+\lambda-\lambda C(z)+\alpha)B_2(s+\lambda-\lambda C(z)+\alpha)]}{s+\lambda-\lambda C(z)+\alpha} \qquad (4.27)$$

Now using equations (4.18) (4.21), (4.23),(4.24),(4.26) and (4.27) in equation (4.17) and solving for $P^{(1)}(0,z,s)$ we get

$$P^{(1)}(0,z,s) = \frac{f_1(z)f_2(z)[1-(s+\lambda-\lambda C(z))V(z,s)]}{DR} \qquad (4.28)$$

Where

$$DR = f_1(z)f_2(z)\{z-(q+pz)B_1(s+\lambda-\lambda C(z)+\alpha)B_2(s+\lambda-\lambda C(z)+\alpha)\} - \alpha\beta z(1-B_1(s+\lambda-\lambda C(z)+\alpha)B_2(s+\lambda-\lambda C(z)+\alpha)) \qquad (4.29)$$

$$f_1(z) = s+\lambda-\lambda C(z)+\alpha \text{ and } f_2(z) = s+\lambda-\lambda C(z)+\beta$$

Substituting the value of $P^{1}(0,z,s)$ from equation (4.28) into equations (4.21), (4.24) & (4.27) we get

$$P^{(1)}(z,s) = \frac{f_2(z)[1-B_1(s+\lambda-\lambda C(z)+\alpha)]}{DR}[1-(s+\lambda-\lambda C(z))V(z,s)] \quad (4.30)$$

$$P^{(2)}(z,s) = \frac{f_2(z)B_1(s+\lambda-\lambda C(z)+\alpha)[1-B_2(s+\lambda-\lambda C(z)+\alpha)]}{DR}[1-(s+\lambda-\lambda C(z))V(z,s)] \quad (4.31)$$

$$R(z,s) = \frac{\alpha z[1-B_1(s+\lambda-\lambda C(z)+\alpha)B_2(s+\lambda-\lambda C(z)+\alpha)]}{DR}[1-(s+\lambda-\lambda C(z))V(z,s)] \quad (4.32)$$

In this section we shall derive the steady state probability distribution for our Queuing model. To define the steady state probabilities, suppress the arguments where ever it appears in the time dependent analysis. By using well known Tauberian property,

$$\lim_{s\to 0} s f(s) = \lim_{t\to\infty} f(t)$$

$$P^{(1)}(z) = \frac{f_2(z)(1-B_1(\lambda-\lambda C(z)+\alpha))}{DR}\lambda(C(z)-1)V(z) \quad (4.33)$$

$$P^{(2)}(z) = \frac{f_2(z)B_1(\lambda-\lambda C(z)+\alpha)(1-B_2(\lambda-\lambda C(z)+\alpha))}{DR}\lambda(C(z)-1)V(z) \quad (4.34)$$

$$R(z) = \frac{\alpha z[1-B_1(\lambda-\lambda C(z)+\alpha)B_2(\lambda-\lambda C(z)+\alpha)]}{DR}\lambda(C(z)-1)V(z) \quad (4.35)$$

In order to determine $P^{(1)}(z)$, $P^{(2)}(z)$, R(z) completely, we have yet to determine the unknown $V(z)$ which appears in the numerator of the right sides of equations (4.33), (4.34) and (4.35). For that purpose, we shall use the normalizing condition.

$$P^{(1)}(1) + P^{(2)}(1) + V(1) + R(1) = 1 \quad (4.36)$$

$$P^{(1)}(1) = \frac{\lambda\beta C'(1)(1-B_1(\alpha))}{dr}V(1) \quad (4.37)$$

$$P^{(2)}(1) = \frac{\lambda\beta C'(1)B_1(\alpha)(1-B_2(\alpha))}{dr}V(1) \quad (4.38)$$

$$R(1) = \frac{\lambda\alpha C'(1)(1-B_1(\alpha)B_2(\alpha))}{dr}V(1) \quad (4.39)$$

Where $dr = \alpha\beta(1-p)B_1(\alpha)B_2(\alpha) - (1-B_1(\alpha)B_2(\alpha))\lambda C'(1)(\alpha+\beta)$

$P^{(1)}(1)$, $P^{(2)}(1)$ and $R(1)$ denote the steady state probabilities that the server is providing first stage of service, second stage of service and server under repair without regard to the number of customers in the queue. Now using equation (4.37), (4.38) and (4.39) into the normalizing condition (4.36) and simplifying, we obtain

$$V(1) = 1 - \frac{\lambda C'(1)}{\beta(1-p)B_1(\alpha)B_2(\alpha)} - \frac{\lambda C'(1)}{\alpha(1-p)B_1(\alpha)B_2(\alpha)} + \frac{\lambda C'(1)}{\beta(1-p)} + \frac{\lambda C'(1)}{\alpha(1-p)} \quad (4.40)$$

and hence, the utilization factor ρ of the system is given by

$$\rho = \frac{\lambda C'(1)}{\beta q B_1(\alpha)B_2(\alpha)} + \frac{\lambda C'(1)}{\alpha q B_1(\alpha)B_2(\alpha)} - \frac{\lambda C'(1)}{\beta q} - \frac{\lambda C'(1)}{\alpha q} \quad (4.41)$$

Where $\rho < 1$ is the stability condition under which the steady states exits.

5. The Mean queue size and the mean system size

Let $P_q(z)$ denote the probability generating function of the queue size irrespective of the server state. Then adding equation (4.27), (4.28) and (4.29) we obtain

$$P_q(z) = P^{(1)}(z) + P^{(2)}(z) + R(z)$$

$$P_q(z) = \frac{N(z)}{D(z)} \quad (5.1)$$

$$N(z) = \lambda(C(z)-1)\left(1 - B_1(\lambda - \lambda C(z) + \alpha)B_2(\lambda - \lambda C(z) + \alpha)\right)(\alpha z + f_2(z))V(z)$$

$$D(z) = f_1(z)f_2(z)\left(z - (q+pz)B_1(\lambda - \lambda C(z) + \alpha)B_2(\lambda - \lambda C(z) + \alpha)\right) - \alpha\beta z(1 - B_1(\lambda - \lambda C(z) + \alpha)B_2(\lambda - \lambda C(z) + \alpha))$$

Let L_q denote the mean number of customers in the queue under the steady state. Then we have

$$L_q = \frac{d}{dz}[P_q(z)] \text{ at } z = 1$$

$$L_q = \lim_{z\to 1} \frac{D'(1)N''(1) - N'(1)D''(1)}{2D'(1)^2} \quad (5.2)$$

where primes and double primes in (4.36) denote first and second derivative at z = 1, respectively. Carrying out the derivative at z = 1 we have

$$\rho = \frac{\lambda C'(1)}{\beta B_1(\alpha)B_2(\alpha)} + \frac{\lambda C'(1)}{\alpha B_1(\alpha)B_2(\alpha)} - \frac{\lambda C'(1)}{\beta} - \frac{\lambda C'(1)}{\alpha}$$

$$N'(1) = \lambda C'(1)(\alpha+\beta)V(1)(1 - B_1(\alpha)B_2(\alpha)) \quad (5.3)$$

$$N''(1) = (1 - B_1(\alpha)B_2(\alpha))\{\lambda C''(1)(\alpha+\beta)V(1) - 2(\lambda C'(1))^2 V(1) + 2\lambda C'(1)\alpha V(1)\} - 2\lambda^2(C'(1))^2(\alpha+\beta)V(1)(B_1(\alpha)B_2'(\alpha) + B_2(\alpha)B_1'(\alpha)) \quad (5.4)$$

$$D'(1) = \alpha\beta B_1(\alpha)B_2(\alpha)(1-p) - (1 - B_1(\alpha)B_2(\alpha))[(\alpha+\beta)\lambda C'(1)] \quad (5.5)$$

$$D''(1) = 2\alpha\beta(1-p)(B_1(\alpha)B_2'(\alpha) + B_2(\alpha)B_1'(\alpha)) - (\alpha+\beta)\lambda C''(1)(1 - B_1(\alpha)B_2(\alpha)) - 2(\alpha+\beta)\lambda C'(1)[1 - pB_1(\alpha)B_2(\alpha) - (B_1(\alpha)B_2'(\alpha) + B_2(\alpha)B_1'(\alpha))] \quad (5.6)$$

Then if we substitute the values from (5.3), (5.4), (5.5) and (5.6) into (5.2) we obtain L_q in the closed form. Further we find the mean system size L using Little's formula. Thus we have

$$L = Lq + \rho$$

where L_q has been found by equation (5.2) and ρ is obtained from equation (4.35)

6. Special case

Case (i): No feedback put p = 0 in the main results, we get

$$\rho = \frac{\lambda C'(1)}{\beta B_1(\alpha) B_2(\alpha)} + \frac{\lambda C'(1)}{\alpha B_1(\alpha) B_2(\alpha)} - \frac{\lambda C'(1)}{\beta} - \frac{\lambda C'(1)}{\alpha}$$

$$N'(1) = \lambda C'(1)(\alpha + \beta) V(1)(1 - B_1(\alpha) B_2(\alpha)) \tag{6.1}$$

$$N''(1) = (1 - B_1(\alpha) B_2(\alpha))\{\lambda C''(1)(\alpha+\beta) V(1) - 2(\lambda C'(1))^2 V(1) + 2\lambda C'(1)\alpha V(1)\} - 2\lambda^2 (C'(1))^2 (\alpha+\beta) V(1)(B_1(\alpha) B_2'(\alpha) + B_2(\alpha) B_1'(\alpha)) \tag{6.2}$$

$$D'(1) = \alpha\beta B_1(\alpha) B_2(\alpha) - (1 - B_1(\alpha) B_2(\alpha))[(\alpha+\beta)\lambda C'(1)] \tag{6.3}$$

$$D''(1) = 2\alpha\beta(B_1(\alpha) B_2'(\alpha) + B_2(\alpha) B_1'(\alpha)) - (\alpha+\beta)\lambda C''(1)(1 - B_1(\alpha) B_2(\alpha)) - 2(\alpha+\beta)\lambda C'(1)[1 - (B_1(\alpha) B_2'(\alpha) + B_2(\alpha) B_1'(\alpha))] \tag{6.4}$$

Then, if we substitute the value $N'(1), N''(1), D'(1), D''(1)$ from equations (6.1) to (6.4) into equation (5.2), we get L_q in the closed form.

7. NUMERICAL RESULTS

For the purpose of a numerical result, we choose the following arbitrary values:

$E(I) = 0.3, E(I/(I-1)) = 0.04, \mu_1 = 4, \mu_2 = 8, \alpha = 5, \beta = 10$ and $p = 0.4$ while λ varies from 0.1 to 1.0 such that the steady state condition is satisfy.

λ	P	V(1)	Lq	Wq	L	W
0.1	0.039844	0.960156	0.009990	0.999040	0.049834	0.498340
0.2	0.079688	0.920313	0.023366	0.116832	0.103054	0.515270
0.3	0.119531	0.880469	0.040632	0.135442	0.160164	0.533879
0.4	0.159375	0.840625	0.623880	0.155971	0.221763	0.554408
0.5	0.199212	0.800781	0.089353	0.178706	0.288572	0.577144
0.6	0.239063	0.760938	0.122397	0.203995	0.361460	0.602432

0.7	0.278906	0.721094	0.162582	0.232600	0.441488	0.630697
0.8	0.318750	0.681240	0. 211218	0.264023	0.529968	0.662461
0.9	0.358594	0.641406	0.269943	0.299937	0.628537	0.698374
1.0	0.398438	0.601562	0.340825	0.340825	0.739263	0.739263

Table 1

The table 1 give computed values of the idle time, the utilization factor, the mean queue size and mean system size of our queueing model. The table I clearly shows as long as, increasing the arrival rate, the server's idle time decreases while the utilization factor, the mean queue size and the mean system size of our queueing model are increases.

8. Conclusion

In this paper we have studied an $M^{[X]}/G/1$ feedback queue with two stage of heterogeneous service subject to breakdown and repair having multiple vacation. The probability generating function of the number of customers in the queue is found using the supplementary variable technique. This model can be utilized in large scale manufacturing industries and communication networks.

9.References

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